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WITH THEIR

SOLUTIONS.

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VOL. XIV.

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WITH MANY

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2023. If n be a prime number,

$$\lfloor n \left\{ \frac{1}{n} - \frac{1}{n-1} + \frac{1}{\lfloor 2\,(n-2)} - \frac{1}{\lfloor 3\,(n-3)} + \ldots\ldots + \frac{1}{\lfloor n-1} - \frac{1}{\lfloor n} \right\}$$

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2111. In a Cartesian oval whose two internal foci coincide, the difference of the two arcs intercepted by any two transversals from the external focus is equal to a straight line which may be found. 21

2151. In a Cartesian oval with a finite node (a nodal Limaçon), the difference of the lengths of the loops is four times the distance between the vertices. 101

2279. An ellipse is described having its major axis three times the minor. Any parabola is drawn satisfying the following conditions:—The intercept made on its principal axis by the axes of the ellipse is constant, and equal to the sum of the semi-axes of the ellipse; its focus lies on the ellipse, in the same

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quadrant as the intercept, and the tangent at its vertex passes through the foot of the perpendicular from the middle of the intercept on the major axis of the ellipse. An equilateral hyperbola is also drawn, whose axis is equal to the intercept, and whose axis and centre fall on the major axis and centre of the ellipse. Now if any tangent is drawn to the parabola, and from the points in which it meets the hyperbola lines are drawn parallel to the asymptotes, one of the loci of their intersections envelopes a circle. 108

2360. Find the average area of all the triangles in which there are given—(1) an angle and the sum of the two containing sides, all possible values of these sides being considered equally probable; (2) the sum of two sides only, all possible values of these two sides, and of the included angle, being considered equally probable; (3) one side and the sum of the other two; (4) the sum of the three sides; all possible values of the sides in (3) and (4) being considered equally probable. Also show (5) that the average of the averages in (3) is seven-eighths of the average in (4). 103

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2596. Two particles are projected with the same velocity, in the same direction, at right angles to the line joining them: find the curves they will describe if they attract according to the law of nature. If the particles were equal, find what law of resistance, in the medium through which they pass, would cause them to describe parts of the same circle. 88

2652. If $l+m+n=0$, prove that

$$(l^2+m^2+n^2)^3-2\,(l-m)^2\,(m-n)^2\,(n-l)^2-54\,l^2m^2n^2$$

vanishes; and hence verify the identity $\Delta = I^3-27J^2$, where Δ is the discriminant, and I, J are the fundamental invariants of $(a, b, c, d, e \between x, y)^4$. 93

2790. Find the mean distance of all the points in a right circular cylinder from one end of the axis. 64

2791. 1. In the sides BC, CA, AB of the triangle ABC points D, E, F are respectively taken at random, and DEF is an inscribed triangle. Again, in the triangles AEF, BFD, CDE, points P, Q, R are respectively taken at random. Show that the average area of the triangle PQR is one-fourth that of ABC.

2. If G_1, G_2, G_3 be the centres of gravity of the triangles AEF, BFD, CDE; then generally $9\Delta\, G_1G_2G_3 = 2\Delta\, ABC + \Delta DEF$. ... 19

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2852. 1. The locus of the centre of circles passing through four points, related as A, B, C, D in Questions 2803, 2833 (*Reprint*, Vol. XII., pp. 37, 90), is the conic polar reciprocal to the given conic with respect to the concentric circle bisecting the intervals between the centre and foci.
2. The normal at B, and the line joining the centre of the circle which osculates the conic at that point with the centre of the circle ABCD, make equal angles with the axis of the conic. 96

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$$\frac{ds}{d\phi} = \frac{u^2}{4g\sin\alpha}\left\{\frac{3\sin\phi + 2\sin^2\phi}{(1+\sin\phi)^2} + \log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right)\right\},$$
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$$\left(\frac{x}{2a}\right)^{\frac{2}{3}} + \left(\frac{y}{2b}\right)^{\frac{2}{3}} = 1. \qquad 87$$

2980. A given curve moves without rotation so as always to touch a fixed curve; in any position draw the common tangent, also a second tangent to the moving curve at right angles to the former: find the envelope of the second tangent. Show that, if the intrinsic equation of the fixed curve be $\rho = f(\theta)$, and that of the moving curve $\rho = \phi(\theta)$, that of the required envelope will be
$$\rho = \frac{d}{d\theta}f(\theta - \tfrac{1}{2}\pi) - \frac{d}{d\theta}\phi(\theta - \tfrac{1}{2}\pi) + \phi(\theta);$$

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3049. A straight line OX touches a conic at O, and from four points P, Q, R, S on the line tangents are drawn to the conic, making angles $\alpha, \beta, \gamma, \delta$ respectively with OX; show that with the usual convention as to sign

$$\frac{\mathrm{OP}\cot\alpha}{\mathrm{PQ}.\mathrm{PR}.\mathrm{PS}}+\frac{\mathrm{OQ}\cot\beta}{\mathrm{QP}.\mathrm{QR}.\mathrm{QS}}+\frac{\mathrm{OR}\cot\gamma}{\mathrm{RP}.\mathrm{RQ}.\mathrm{RS}}+\frac{\mathrm{OS}\cot\delta}{\mathrm{SP}.\mathrm{SQ}.\mathrm{SR}}=0.$$ 105

3050. If a, b, c be the three sides of the triangle determined on the surface of a sphere by the centres of any three small circles of the sphere; A, B, C its three angles; p, q, r the spherical radii of the three circles; and k that of the circle orthogonal to the three: prove that

$$\sec^2 k=\frac{l^2+m^2+n^2-2mn\cos\mathrm{A}-2nl\cos\mathrm{B}-2lm\cos\mathrm{C}}{4\sin s\sin(s-a)\sin(s-b)\sin(s-c)};$$

where $l=\cos p\sin a$, $m=\cos q\sin b$, $n=\cos r\sin c$. 36

3053. Prove that, if

$$\left(\frac{4q^{\frac{1}{2}}}{kk'}\right)^{\frac{1}{6}}\left(\frac{\pi}{2\mathrm{K}}\right)^{\frac{1}{2}}.\,\Theta(q,x)=(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)\ldots,$$

then

$$\sin am\left\{\frac{2(1+k)\mathrm{K}x}{\pi},\frac{2k^{\frac{1}{2}}}{1+k}\right\}=\left(\frac{(1+k)\pi}{2\mathrm{K}}\right)^{\frac{1}{2}}.\frac{\Theta^2(q,x)}{\Theta(q^{\frac{1}{2}},x)}.\sin am\frac{2\mathrm{K}x}{\pi}.$$ 56

3074. To prove that

$$\Gamma(n+1)=n+a_1 n(n-1)+a_2 n(n-1)(n-2)+\ldots$$
$$\ldots+a_r n(n-1)\ldots(n-r)+\ldots,$$

where $a_r=1-1+\frac{1}{1.2}-\frac{1}{1.2.3}+\ldots\pm\frac{1}{1.2\ldots r}$,

that is, $=$ the sum of the first $r+1$ terms of the expansion of ϵ^{-1}. 64

3077. A′, B′, C′ are the middle points of the sides of the triangle ABC. Prove (1) that the line joining A′ to the pole of B′C′ with respect to the inscribed circle passes through the centre of the circle escribed in the angle A. Also (2) verify Faure's theorem that the triangle formed by the polars of A′, B′, C′ is equal in area to ABC. 60

3087. If A, B, C be the three spherical centres of any three coaxal circles on the surface of a sphere; a, b, c their three spherical radii; and α, β, γ their three angles of intersection with any arbitrary circle of the sphere: prove the following general relation

$$\sin\mathrm{BC}\sin a\cos\alpha+\sin\mathrm{CA}\sin b\cos\beta+\sin\mathrm{AB}\sin c\cos\gamma=0.$$ 35

3088. The condition that the shortest distances between the opposite edges of a tetrahedron shall meet in a point, is either that the bisectors of the edges shall be at right angles to one another, or else that

$$p^2(-\sin^2\alpha+\sin^2\beta+\sin^2\gamma)=q^2(\sin^2\alpha-\sin^2\beta+\sin^2\gamma)$$
$$=r^2(\sin^2\alpha+\sin^2\beta-\sin^2\gamma),$$

where p, q, r are the bisectors, and α, β, γ the angles which they make with one another. 25

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3090. A portion of the water in a pond, in the shape of a vertical cylinder, radius a, is rotating about its axis, the angular velocity at any point being $k(a^n - r^n)^{\frac{1}{2}}$, where r is the distance from the axis. Find the form of the free surface, and show that the volume of the depression below the surface of the pond is $\dfrac{\pi k^2 n a^4}{4g(n+4)}$. 85

3095. Given three angles formed by inclined straight lines: prove that the locus of a point such that from it can be drawn three right lines cutting off equal intercepts from corresponding arms of the angles, is a conic. 72

3096. For the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$,

$$\frac{\text{the product of the four normals from } (x, y)}{\text{the product of the two tangents from } (x, y)} = \frac{b^2x^2 + a^2y^2}{a^2 - b^2}.$$

Hence deduce Mr. Wolstenholme's elegant theorem:—"The product of the three normals to a parabola, divided by the product of the two tangents, is constant, and equal to one-fourth of the latus rectum." 38

3097. Show that at any point on an equiangular spiral the conic of closest contact is an ellipse, and the point of contact is at the extremity of one of the equi-conjugate diameters of the ellipse, and the tangent of the angle between these diameters is three times the tangent of the angle of the spiral. 20

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3103. To prove that

$$\frac{1}{\Gamma(n+1)} = 1 - \frac{1}{2}\frac{n(n-1)}{1.2} + \frac{2}{3}\frac{n(n-1)(n-2)}{1.2.3} - \ldots\ldots$$

$$\ldots\ldots + (-1)^r a_r \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r},$$

where $a_r = 1 - \dfrac{r}{1^2} + \dfrac{r(r-1)}{(1.2)^2} - \dfrac{r(r-1)(r-2)}{(1.2.3)^2} + \ldots$ 64

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$\int x^m (a + bx^n)^p\, dx$. 77

3115. To find five biquadrate numbers whose sum is a biquadrate number. 86

3118. If, in a spherical triangle, $A = \frac{2}{5}\pi$, $B = \frac{1}{3}\pi$, $C = \frac{1}{2}\pi$, prove that (1) $\tan c = 2$, $a + b = \frac{1}{2}\pi$, $c = 2b$; (2) the median arc from C to AB is $\frac{1}{5}\pi$, and the perpendicular arc is $\frac{1}{6}\pi$; (3) $\tan \frac{1}{2}a \tan \frac{1}{2}b = \tan \frac{1}{20}\pi$; (4) $\tan r \tan r_1 \tan r_2 \tan r_3 = \frac{1}{20}$. 81

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3119. A rectangle containing mn squares, p black and the rest white, is divided into n rows and m columns, containing respectively m and n squares each. It is proposed to arrange the black squares so that $a_0, a_1, a_2 \dots a_m$ rows shall contain respectively $0, 1, 2 \dots m$ black squares each, and at the same time $b_0, b_1, b_2 \dots b_n$ columns respectively $0, 1, 2 \dots n$ each. Find the necessary and sufficient relations between the a's and the b's for the possibility of such an arrangement, and point out a method of arranging which shall of itself test this possibility. 22

3120. To find the equation of the Jacobian of the quadric surface through the six points

$(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 1, 1), (\alpha, \beta, \gamma, \delta)$. 67

3128. Prove that, X being any function of x,

$$f'(\mathrm{D})\,\mathrm{X} = \left\{f(\mathrm{D})\,x - xf(\mathrm{D})\right\}\mathrm{X};$$

also that $$\phi\left(\frac{d}{d\mathrm{D}}\right)f(\mathrm{D})\,\mathrm{X} = \phi\,(x_2 - x_1)\,f(\mathrm{D})\,\mathrm{X},$$

where x_1 stands for x *preceding* $f(\mathrm{D})$, x_2 for x *following* $f(\mathrm{D})$; the meaning being that, if $\phi\,(x_2 - x_1)$ is expressed by expansion or otherwise in a series of terms such as $u_{x_1} v_{x_2}$, the value of the symbolical expression above will consist of a corresponding series of terms such as $u_x\, f(\mathrm{D})\, v_x\,\mathrm{X}$.

As an instance, $e^{h\frac{d}{d\mathrm{D}}} f(\mathrm{D})\,\mathrm{X} = e^{-hx} f(\mathrm{D})\, e^{hx}\,\mathrm{X}$. 41

3131. In a rectangle of mn squares, divided regularly into n rows and m columns, p black squares are placed in a specified manner: find the probability that a rook, placed on any square at random, may move without changing colour; also the minimum and maximum value of the probability for different arrangements of the p black squares. 78

3132. A mark is made on a vertical tower at a known height from the horizontal plane; the altitude of this mark and of the top of the tower is observed from a point in the plane; find the probable error of the height of the tower deduced from these observations, in terms of the probable error in the measurement of an angle; and show that the best position of observation is that at which the sum of the two altitudes is a right angle. ... 47

3133. Given two small circles of a sphere, one of which (A) passes through the centre of the other (B). An arc of a great circle moves on the sphere with one extremity on (A) and its middle point on (B), and its length is equal to the diameter arc of (B). The locus of the other extremity is represented by the equation

$$(\cos^2 \tfrac{1}{2}\rho - \cos^2 r)\,(\cos\alpha\cos\tfrac{1}{2}\rho - \sin\alpha\sin\tfrac{1}{2}\rho\cos\theta)^2 - \sin^2\alpha\cos^2 r\sin^2\theta = 0,$$

together with the small circle $\cos^2 \frac{1}{2}\rho - \cos^2 r = 0$ (the centre of B being the pole). If we only include quantities up to the second order dependent on the spherical form of the surface, what is the nature of the approximately plane curve traced? 110

3136. Inside a cylindrical bucket, partly filled with water, floats another. Enough water is poured into the outer one to fill it. Had the same quantity been poured into the inner one, the

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outer one would still have been filled. If the descent of the inner bucket in the second case be equal to its rise in the first, show that the ratio of the radii of the buckets is $\sqrt{2}$. 30

3137. Find three square numbers in arithmetical progression, such that, if from each its root be *subtracted*, the three remainders shall be rational squares. 54

3141. Through the focus of an ellipse two chords are drawn at right angles to each other. Find (1) the average, (2) the maximum, and (3) the minimum area of the quadrilateral formed by joining their extremities. 61

3144. If the extremities A, A′ of a given line AA′ describe given lines respectively, show that there is a point rigidly connected with AA′ which describes a circle. 19

3148. Referring to Prof. Cayley's Quest. 3126, it may be shown that

$$x : y : z = (a^3+b^3+c^3)(b-c)-(b^3-c^3)\,a$$
$$: (a^3+b^3+c^3)(c-a)-(c^3-a^3)\,b$$
$$: (a^3+b^3+c^3)(a-b)-(a^3-b^3)\,c$$

is also a point on the curve. What is the geometrical connexion between this and the point $x : y : z = a : b : c$? 52

3150. Across a valley is thrown a dam, converting it into a reservoir, whose surface varies as the square of the depth. Down the valley flows a stream of given velocity and section. If in the bottom of the dam be left a tunnel of small given section, find how long the dam will take to fill to a certain depth. 56

3151. In a Cartesian oval with a finite node (a nodal Limaçon), the difference of the lengths of the loops is four times the distance between the vertices. 21

3153. A conic circumscribes a quadrilateral ABCD; E is the pole of AB, F that of CD; and AB, CD intersect in O. Through E is drawn a straight line cutting the conic in Q, R, and CD in M; and upon this line a point P is taken to make the range (QPRM) harmonic. Show that the locus of P will be a conic AEBCFD, the tangents to which at E, F pass through O; and that the tangents from O to the first conic will pass through the four points in which the common tangents to the two conics touch the second. .... 34 & 80

3155 If the angular escribed radii of a spherical triangle be r_1, r_2, r_3, and if $r_1+r_2+r_3=\pi$, show that

$$2\tan R \tan s \tan S = \cos s;$$

and if, at the same time, $S=\pi$, then

$$R_1+R_2+R_3 = \tfrac{1}{2}\pi. \text{ .... } 30$$

3157. The circle inscribed in ABC touches BC in D: prove that the trilinear equation to the straight line joining the middle point of AD to middle point of BC is $(b-c)\,\alpha = b\beta - c\gamma$. 58

3158. It is required to find n^3 consecutive numbers of the natural series, the sum of whose cubes shall be a cube. 32

3159. Find general integral values for x, y, z that will make

$$xy-1 = \square,\quad xz-1 = \square,\quad yz-1 = \square \text{ .... } 75$$

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3161. A ladder rests against a vertical wall, to which it is inclined at an angle of 45°; the coefficients of friction of the wall and horizontal plane being respectively $\frac{1}{3}$ and $\frac{1}{2}$, and the centre of gravity of the ladder being at its middle round. A man whose weight is half the weight of the ladder begins to ascend it: find to what height he will go before the ladder begins to slide. 71

3168. It is required to make the lower side of a rocker, which is loaded equally at both ends, of such a form that, when rolling on a horizontal platform, one end will have a uniform preponderance over the other. 73

3169. 1. Let U be a Representative Quantity which satisfies the equation $U(U-1)(U-2)\ldots(U-n+1) = 1$;

$$\text{then } U^n (=U_n) = \left(\frac{\Delta}{1} + \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} + \ldots\ldots + \frac{\Delta^n}{1.2\ldots n}\right) 0^n.$$

2. Let $U(U+1)\ldots(U+n-1) = 1$, then

$$U^n (=U_n) = (-1)^{n+1} \left(\frac{\Delta}{1} - \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} - \ldots\ldots \pm \frac{\Delta^n}{1.2\ldots n}\right) 0^n.$$ 106

3174. A radius vector is drawn from a given point in the circumference of a circle, and a circle described on it as diameter: find the average area common to both circles. 53

3177. An ellipse and an hyperbola are confocal, and at the common points the tangents to the ellipse are parallel to the asymptotes of the hyperbola: show that the axes of the ellipse are in the duplicate ratio of those of the hyperbola. 61

3179. Given the perimeter and an angle of a triangle, prove that the area is a maximum when the sides containing the given angle are equal. 81

3183. Find three integral numbers in geometrical progression, such that if each be increased by unity the sums will be squares. ... 95

3186. Prove that the perpendicular to the line bisecting the vertical angle of a triangle, drawn from the point in which the bisector meets the base, intercepts on either side a length which, measured from the vertex, is equal to the harmonic mean between the sides. 76

3187. A chord is drawn at random across a given circle, and then two chords are taken at random in the surface. Find the chance that both points are on the same side of the chord. 71

3197. If the epicycloid described by a point on the circumference of a circle rolling on an equal fixed circle be loaded with matter proportional to its curvature at every point, the centre of gravity of the whole will be at the centre of the fixed circle. 98

3198. A polished wire of circular section bent into the form of a circle rolls rapidly on the outside of a fixed circle in the same horizontal plane which is exposed to the rays of the sun. Show that a spectator at a considerable distance on the side remote from the sun will see a bright curve consisting of two equal circles touching the fixed circle. 70

3204. Find the average distance of the right angle of a given right-angled triangle from the hypotenuse. 84

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3205. Solve the linear partial differential equation of the second order

$$2(xr-t)+mp=0 \text{ ....(1)}$$

for the cases in which $m=1$ and $m=3$. 66

3207. If $(a_1, a_2, \ldots a_n)$, $(b_1, b_2, \ldots b_n)$ be two sets of quantities all distinct from one another; show that, in order to the co-existence of the $2n$ equations of the form

$$(a_i-b_1)(a_i-b_2) \ldots\ldots (a_i-b_n) = A_i (a_i-a_1)(a_i-a_2)\ldots\ldots$$

$$(b_i-a_1)(b_i-a_2) \ldots\ldots (b_i-a_n) = B_i (b-b_1)(b_i-b_2)\ldots\ldots,$$

we must have $\Sigma_n^1 A_i + \Sigma_n^1 B_i = 0$. 68

3210. At a chess tournament the players are supposed to be divisible into two classes, the odds on a member of the first in a game with a member of the second being 2 to 1. The second class is twice as numerous as the first. A player is observed to win a game, and a bet of 7 to 10 is made that he belongs to the first class. Show that there are 18 players engaged. 73

3212. A cube just filled with homogeneous fluid is made to rotate about an axis through its centre of gravity. Show that the whole pressure on its surface is independent of what axis be chosen, and bears to the pressure on a sphere of the same surface, filled with similar fluid and rotating with the same velocity, the ratio $10\pi : 36$. 83

3213. If three circles intersect two and two in the points A, A′; B, B′; C, C′; and through any other point O circles OAA′, OBB′, OCC′ be described, these will all meet again in another point. 102

3215. Through two points taken at random on the circumference of a given circle, lines are drawn in random directions: find the chance that they will intersect within the given circle. 72

3221. Determine the condition that a straight line and two conics may have two common points of intersection. 91

3224. From any point T on a straight line OX, TQ is drawn making a constant angle with the polar of T with respect to a fixed conic: show that the envelope of TQ is in general a parabola, which reduces to a point if the constant angle be that which OX makes with the direction conjugate to OX. 83

3244. Through any point P, within or without a parabola whose focus is S, a double ordinate QQ′ is drawn; the polar of P cuts the axis in M; the perpendicular from P upon this polar meets it in N and the axis in R: show that M, N, Q, R, Q′ all lie on a circle whose centre is S. 83

3249. Given on a given conic two quadrangles PQRS and *pqrs*, having the same centres, and such that P, *p*; Q, *q*; R, *r*; S, *s* are the corresponding vertices (that is, the four lines PQ, RS, *pq*, *rs*, all pass through the same point; and similarly the lines PR, QS, *pr*, *qs*, and the lines PS, QR, *ps*, *qr*): it is required to show that a conic may be drawn through the points *p*, *q*, *r*, *s*, touched at these points by the lines *p*P, *q*Q, *r*R, *s*S, respectively. 104

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3258. Two points are taken at random in a given radius of a circle, and a third point anywhere on the surface of the circle. Find the chance that the triangle formed by joining these points is acute. 97

3268. ABC, A'B'C' are two straight lines so divided that AB : BC = A'B' : B'C'; show that parallels through B, B' to AA', CC' meet on AC', A'C. 109

CORRIGENDA.

VOL. X.

Quest. 2577, pp. x. and 84, line 3, *for* ellipse *read* semicircle.

VOL. XII.

p. 67, line 12, *for* $\int_0$ *read* $\int_0^{\frac{1}{2}\pi}$.

p. 98, line 9, *for* r^{-n} *read* r_p^{-n}.

p. 105, *eq.* (5), *for* $+a_{n-1}$ *read* $+a_{n-2}$, *and for* $+b_{n-1}$ *read* $+b_{n-2}$;

„ line 24, *for* $1^2+0^\circ=1^\circ$ *read* $1^2+0^2=1^2$;

„ line 33, *for* $\dfrac{2}{2\sqrt{2}}$ *read* $\dfrac{1}{2\sqrt{2}}$.

p. 106, *eq.* (C), *for* $+x_{2p+1}$ *read* x_{2p-1}.

p. 108, line 2, *for* fi *read* figure.

VOL. XIII.

p. 29, line 16, *for* $-\dfrac{6783}{262144}$ *read* $=\dfrac{6783}{262144}$.

VOL. XIV.

p. 32, Quest. 3158. Mr. Evans here supposes that the numbers begin with *unity*. On another hypothesis, the problem is shown to be possible, and solutions are given, in Vol. XV. of the *Reprint*.

p. 72, line 8, *for* $\dfrac{4}{y}$ *read* $\dfrac{4}{7}$.

p. 84, line 4 from bottom, *for* BP *read* CP;

„ line 3 „ *for* CD *read* BD;

„ last line, *for* log (...) *read* $\log\dfrac{(a+c)(b+c)}{ab}$.

p. 97, line 5 from bottom, *for* π *read* $\frac{1}{4}\pi$;

„ last line, *for* $\frac{1}{3}\pi$ *read* $\frac{1}{12}\pi$, *and for* $\frac{5}{12}$ *read* $\frac{7}{24}$.

p. 100, line 9, *add* [by putting $y=\tan\theta$].

p. 101, line 6, *for* A′B′C *read* A′B′C′.

p. 102, line 26, *for* form *read* focus.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

3002. (Proposed by MATTHEW COLLINS, B.A.)—If every two of five circles A, B, C, D, E touch each other, except D and E, prove that the common tangent of D and E is just twice as long as it would be if D and E touched each other.

Solution by Professor CAYLEY.

Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, foci R, S; the coordinates of a point U on the ellipse may be taken to be $(a\cos u,\ b\sin u)$, and then the distances of this point from the foci will be

$$r = a(1-e\cos u),\quad s = a(1+e\cos u).$$

Taking k arbitrarily, with centre R describe a circle radius $a-k$, with centre S a circle radius $a+k$, and with centre U a circle radius $k-ae\cos u$: say these are the circles R, S, U respectively; the circle U will touch each of the circles R, S (viz., assuming $k>ae<a$, so that the foregoing radii are all positive, it will touch the circle R externally and the circle S internally).

Considering next a point V, coordinates $(a\cos v,\ b\sin v)$, and the circle described about this point with the radius $k-ae\cos v$, say the circle V; this will touch in like manner the circles R, S respectively. And the circles U, V may be made to touch each other externally; viz., this will be the case if squared sum of radii = squared distance of centres, or what is the same thing, squared difference of radii + 4 times the product of radii = squared distance of centres; that is,

$$a^2e^2(\cos u-\cos v)^2+4(k-ae\cos u)(k-ae\cos v)$$
$$= a^2(\cos u-\cos v)^2+b^2(\sin u-\sin v)^2,$$

or $$2(k-ae\cos u)(k-ae\cos v) = b^2\{1-\cos(u-v)\}.$$

If for a moment we write $\tan\frac{1}{2}u = x$, $\tan\frac{1}{2}v = y$, and therefore

$$\cos u = \frac{1-x^2}{1+x^2},\quad \cos v = \frac{1-y^2}{1+y^2},\quad \sin u = \frac{2x}{1+x^2},\quad \sin v = \frac{2y}{1+y^2},$$

$$\cos(u-v) = \frac{(1-x^2)(1-y^2)+4xy}{(1+x^2)(1+y^2)},\quad 1-\cos(u-v) = \frac{2(x-y)^2}{(1+x^2)(1+y^2)},$$

we have $$\left(k - \frac{ae(1-x^2)}{1+x^2}\right)\left(k - \frac{ae(1-y^2)}{1+y^2}\right) = \frac{b^2(x-y)^2}{(1+x^2)(1+y^2)},$$

or $$\{k-ae+(k+ae)x^2\}\{k-ae+(k+ae)y^2\} = b^2(x-y)^2,$$

which is readily identified with the circular relation

$$\tan^{-1} y\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}} - \tan^{-1} x\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}} = \tan^{-1}\left(\frac{k^2-a^2e^2}{a^2-k^2}\right)^{\frac{1}{2}}:$$

or, what is the same thing, in order that the circles U, V may touch, the relation between the parameters u, v must be

$$\tan^{-1}\left\{\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}v\right\} - \tan^{-1}\left\{\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}u\right\} = \tan^{-1}\left(\frac{k^2-a^2e^2}{a^2-k^2}\right)^{\frac{1}{2}}.$$

Considering in like manner a circle, centre the point W, coordinates ($a\cos w$, $b\sin w$), and radius $k-ae\cos w$, say the circle W; this will touch as before the circles R, S; and we may make W touch each of the circles U, V; viz., we must have

$$\tan^{-1}\left\{\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}w\right\} - \tan^{-1}\left\{\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}\tan^{-1}\tfrac{1}{2}v\right\} = \tan^{-1}\left(\frac{k^2-a^2e^2}{a^2-k^2}\right)^{\frac{1}{2}},$$

$$\tan^{-1}\left\{\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}u\right\} - \tan^{-1}\left\{\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}\tan^{-1}\tfrac{1}{2}w\right\} = \tan^{-1}\left(\frac{k^2-a^2e^2}{a^2-k^2}\right)^{\frac{1}{2}},$$

where, in the last equation, $\tan^{-1}\left\{\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}\tan\frac{1}{2}u\right\}$ must be considered as denoting its value in the first equation increased by π. Hence, adding the three equations, we have

$$\pi = 3\tan^{-1}\left(\frac{k^2-a^2e^2}{a^2-k^2}\right)^{\frac{1}{2}},\quad \text{that is}\quad \left(\frac{k^2-a^2e^2}{a^2-k^2}\right)^{\frac{1}{2}} = \tan\tfrac{1}{3}\pi = \sqrt{3};$$

or $$k^2-a^2e^2 = 3(a^2-k^2),\quad \text{that is}\quad 3a^2-4k^2+a^2e^2 = 0;$$

viz., this is the condition for the existence of the three circles U, V, W, each touching the two others, and also the circles R, S.

The circle R lies inside the circle S, and the tangential distance is thus imaginary; but defining it by the equation

squared tangential dist. = squared dist. of centres − squared sum of radii,

the squared tangential distance is $= 4a^2e^2-4a^2$.

But if the circles were brought into contact, the distance of the centres would be $2k$, and the value of the squared tangential distance $= 4k^2-4a^2$; hence, if this be = one-fourth of the former value, we have

$$4(k^2-a^2) = a^2e^2-a^2,\quad \text{that is}\quad 3a^2-4k^2+a^2e^2 = 0,$$

the same condition as above. The solution might easily be varied in such

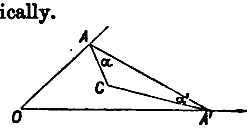
wise that the circles R, S should be external to each other, and therefore the tangential distance real; but the case here considered, where the locus of the centres of the circles U, V, W is an ellipse, is the more convenient, and may be regarded as the standard case.

3144. (Proposed by Professor CAYLEY.)—If the extremities A, A′ of a given line AA′ describe given lines respectively, show that there is a point rigidly connected with AA′ which describes a circle.

Solution by T. COTTERILL, M.A.; Rev. R. TOWNSEND, M.A., F.R.S.; C. W. MERRIFIELD, F.R.S.; Rev. J. WOLSTENHOLME, M.A.; *and others.*

Let the lines meet in O. Then, since the side AA′ and the angle O remain the same, the radius of the circle circumscribing the triangle OAA′ in its different positions is constant, and the centre C of this circle (supposed to be rigidly connected with AA′ by the radii through them) describes a circle centre O. Any fixed point in the circumference of this circle describes an indefinitely thin ellipse, each half being a diameter of the circle.

These results are also easily obtained analytically.



For, let x, y be the perpendiculars from any point c (rigidly connected with AA′ by $c\text{A} = a$, $c\text{A}' = a'$) on the lines OA and OA′ respectively. Let CAA′ $= \alpha$, CA′A $= \alpha'$, OAA′ $= \theta$, OA′A $= \theta'$, and AOA′ $= \omega$, ACA′ $= \gamma$.

Then $$x = a\sin(\theta-\alpha), \text{ and } y = a'\sin(\theta'-\alpha').$$

But $\sin^2(\theta-\alpha+\theta'-\alpha') =$
$$\sin^2(\theta-\alpha)+\sin^2(\theta'-\alpha')+2\sin(\theta-\alpha)\sin(\theta-\alpha')\cos(\theta+\theta'-\alpha-\alpha').$$

Hence $$\sin^2(\gamma-\omega) = \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+2\frac{xy}{ab}\cos(\gamma-\omega),$$

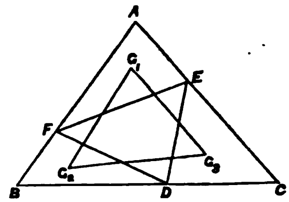
also $$\sin^2\omega\,(\text{OC})^2 = x^2+y^2+2xy\cos\omega.$$

From these two equations the result given above at once follows.

2791. (Proposed by STEPHEN WATSON.)—1. In the sides BC, CA, AB of the triangle ABC points D, E, F are respectively taken at random, and DEF is an inscribed triangle. Again, in the triangles AEF, BFD, CDE points P, Q, R are respectively taken at random. Show that the average area of the triangle PQR is one-fourth that of ABC.

2. If G_1, G_2, G_3 be the centres of gravity of the triangles AEF, BFD, CDE; then generally $9\triangle G_1G_2G_3 = 2\triangle ABC + \triangle DEF$.

Solution by the PROPOSER.



1. If two of the points, as P, Q, remain fixed while R takes every position in its triangle, the average area of the triangle PQR will be that when R is at the centre of gravity of the triangle CDE. From this it follows that the average area of the triangle PQR, when P, Q, R take every position in their respective triangles, is the triangle $G_1G_2G_3$.

Take BC, BA for axes of x and y; put $BD = ax$, $CE = by$, $AF = cz$; and denote G_1, G_2, G_3 by x_1y_1, x_2y_2, x_3y_3 respectively. Then if $\triangle ABC = m$,

$$\triangle AEF = z(1-y)\,m, \quad \triangle BFD = x(1-z)\,m, \quad \triangle CDE = y(1-x)\,m \quad \text{......(1)}.$$

Also $\qquad x_1 = \tfrac{1}{3}a(1-y), \quad x_2 = \tfrac{1}{3}ax, \quad x_3 = \tfrac{1}{3}a(2+x-y),$

and $\qquad y_1 = \tfrac{1}{3}c(2+y-z), \quad y_2 = \tfrac{1}{3}c(1-z), \quad y_3 = \tfrac{1}{3}cy,$

$$\therefore \triangle G_1G_2G_3 = \tfrac{1}{2}\sin B\left\{(y_1+y_2)(x_1-x_2)+(y_2+y_3)(x_2-x_3)+(y_3+y_1)(x_3-x_1)\right\}$$

$$= \tfrac{1}{9}m\left\{3-x(1-z)-y(1-x)-z(1-y)\right\}, \text{ which by (1)}$$

$$= \tfrac{1}{3}m-\tfrac{1}{9}(m-\triangle DEF) = \tfrac{2}{9}m+\tfrac{1}{9}\triangle DEF \quad \text{........................(2)}.$$

When D, E, F take all positions on BC, CA, AB, the average area of $\triangle DEF = \tfrac{1}{4}m$; hence the average area of $\triangle G_1G_2G_3 = \tfrac{1}{4}m$, and this is the average area of the triangle PQR required.

2. From (2) we have $9\triangle G_1G_2G_3 = 2m + \triangle DEF$.

3097. (Proposed by the Rev. W. A. WHITWORTH, M.A.)—Show that at any point on an equiangular spiral the conic of closest contact is an ellipse, and the point of contact is at the extremity of one of the equi-conjugate diameters of the ellipse, and the tangent of the angle between these diameters is three times the tangent of the angle of the spiral.

Solution by the Rev. J. WOLSTENHOLME, M.A.

A conic can always be drawn having contact of the fourth order with a given curve at a given point; and to have such contact it is necessary and sufficient that ρ, $\dfrac{d\rho}{ds}$, $\dfrac{d^2\rho}{ds^2}$ have the same values in the two curves at the point of contact, ρ being the radius of curvature and s the arc. Now, in the equiangular spiral, $\quad \dfrac{d\rho}{ds} = \cot\alpha, \quad \dfrac{d^2\rho}{ds^2} = 0.$

In a conic, $$\rho = \frac{a^2b^2}{(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{3}{2}}},$$

ϕ being the angle which the tangent makes with the major axis $2a$;

hence, in the conic, $\dfrac{d\rho}{ds} = -\dfrac{3(a^2-b^2)\sin\phi\cos\phi}{a^2\sin^2\phi+b^2\cos^2\phi} \equiv -\dfrac{3(a^2-b^2)}{a^2\tan\phi+b^2\cot\phi}$;

and in order that $\dfrac{d^2\rho}{ds^2}=0$, we must have $\tan^2\phi=\dfrac{b^2}{a^2}$, which proves that the conic is an ellipse, and the contact at the end of one of the equi-conjugate diameters. Also

$$\cot\alpha = -\frac{3(\cos^2\phi-\sin^2\phi)\sin\phi\cos\phi}{2\sin^2\phi\cos^2\phi} \equiv -3\cot 2\phi,$$

hence the tangent of the acute angle between the equi-conjugate diameters is three times the tangent of the acute angle of the spiral. [Much of this was set in St. John's College many years ago, and appears as an Example in Todhunter's *Differential Calculus*.]

3151. (Proposed by S. ROBERTS, M.A.)—In a Cartesian oval with a finite node (a nodal Limaçon), the difference of the lengths of the loops is four times the distance between the vertices.

2111. (Proposed by M. W. CROFTON, F.R.S.)—In a Cartesian oval whose two internal foci coincide, the difference of the two arcs intercepted by any two transversals from the external focus is equal to a straight line which may be found.

Solution by SAMUEL ROBERTS, M.A.

These two theorems may be treated together. If the Cartesian is represented by $\rho^2 \mp 2A\rho - 2Bx + C^2 = 0$, the condition for a double point not at the origin is $C^2 = (A-B)^2$. If $A>B$, the Limaçon has a node; if $B>A$, it has a conjugate point.

For the difference of the elements of corresponding arcs we have

$$2\sqrt{A^2+B^2-(A-B)^2+2AB\cos\omega}\,.\,d\omega = 4\sqrt{AB}\cos\tfrac{1}{2}\omega\,.\,d\omega.$$

Integrating between ω_2, ω_1, we get $8\sqrt{AB}\,(\sin\tfrac{1}{2}\omega_2-\sin\tfrac{1}{2}\omega_1)$.

But since $\omega=0$ gives $\rho = A+B\pm 2\sqrt{AB}$, the distance between the vertices is $4\sqrt{AB}$; therefore, for a nodal Limaçon, we find, by making $\omega_2 = 180°$, $\omega_1 = 0$, and doubling the result, that the difference of the lengths of the loops is four times the distance between he vertices.

In the case of a conjugate point, the corresponding arcs lie on the oval. Since $\sqrt{AB}$ and (ω_2, ω_1) are known, we can, of course, determine a line o the required value.

The result relative to the nodal Limaçon is remarkable, since, if the curve is represented by $\rho = \mp a + b\cos\omega$, the difference in question depends on a only, and is therefore independent of the magnitude of the circle by aid of which the curve may be described.

3119. (Proposed by MORGAN JENKINS, M.A.)—A rectangle containing mn squares, p black and the rest white, is divided into n rows and m columns, containing respectively m and n squares each. It is proposed to arrange the black squares so that $a_0, a_1, a_2, \ldots a_m$ rows shall contain respectively $0, 1, 2, \ldots m$ black squares each, and at the same time $b_0, b_1, b_2, \ldots b_n$ columns respectively $0, 1, 2, \ldots n$ each. Find the necessary and sufficient relations between the a's and the b's for the possibility of such an arrangement, and point out a method of arranging which shall of itself test this possibility.

Solution by the PROPOSER.

From the given data the four equations

$$\left.\begin{array}{r} a_0+a_1+a_2+\ldots\ldots+a_m \quad = n \ldots(1) \\ a_1+2a_2+\ldots\ldots+m\,.\,a_m = p \ldots(2) \end{array}\right\} \quad \left.\begin{array}{r} b_0+b_1+b_2+\ldots\ldots\ldots b_n = m \ldots(3) \\ b_1+2b_2+\ldots\ldots\ldots+nb_n = p \ldots(4) \end{array}\right\}$$

are obviously necessary; but before finding the additional conditions of inequality which must be satisfied, I will describe a method of arranging which must succeed if with the given values of the a's and b's an arrangement be possible.

For convenience of description, suppose the black squares to be of uniform weight and the white squares without weight. Arrange first the row of greatest weight, and so on in descending order of weight; always placing the black squares in those columns which have the greatest weight in the rectangle formed by excluding the previously arranged rows, subject also, of course, to the condition that the total weight of a column is not to be exceeded. We might also begin with columns instead of rows, and, mutatis mutandis, proceed in ascending order of weight.

The working of the method may be further facilitated thus:—It is evident that, since in any possible arrangement any two rows or columns may be interchanged, the original rows and columns may be supposed to be placed in order of weight. Suppose, then, that in the rectangle OAPB the rows are placed in descending order of weight from OA to BP, and the columns from OB to AP. Start with the row at AO and place the black squares as far as possible towards O, subject to the obvious condition beforenamed, and also to the modification that, so far as there is any choice amongst columns of equal weight, the black squares are to be placed as far as possible away from O.

For proof of the method, as first described, we see that if in the row AO there were a white and a black square, the white being in a column of equal or greater weight than the black, then in some row below the positions must be reversed; thus $\begin{smallmatrix}\mathrm{BW}\\ \mathrm{WB}\end{smallmatrix}$; and by interchanging B and W in the small square we could, without altering the weight either of rows or columns of the rectangle, force the black square in AO into the column of greater weight, or change it about at pleasure amongst columns of equal weight; and therefore we may place the black squares in the row AO in the specified manner without affecting the possibility of arrangement.

In like manner for the next row, and so on; which proves what is required.

The second method is seen to be merely a simple way of working the first, the use of the modification being to keep the order of weight of the columns of the remaining rectangle the same as in the whole rectangle.

It may be noticed that all arrangements which give the same weights to the respective rows and columns give the same centre of gravity of the whole rectangle, and that when the rows and columns are both in descending order of weight the distance of the centre of gravity from O is a minimum.

Exs. Let the a's in descending order of magnitude of their weight be

$$1, 1, 2, 1, 1, 0,$$

and the b's in descending order of magnitude of their weight be

$$1, 1, 1, 0, 1, 1, 0;$$

then these sets of values satisfy the four equations of the form (1), (2), (3), (4); because the sum of the digits in each set is one less than the number of digits in the other set, and the total weight of each set,—that is, the number represented by each set when to each digit is given the local value by multiplying the digits respectively by 0, 1, 2......&c., from the right-hand side—is the same.

Thus $1 . 5 + 1 . 4 + 2 . 3 + 1 . 2 + 1 . 1 = 1 . 6 + 1 . 5 + 1 . 4 + 1 . 2 + 1 . 1 = 18.$

Here $p = 18$, $m = 6$, $n = 5$. So we may take 1, 2, 1, 0, 2, 0 instead of 1, 1, 2, 1, 1, 0 for the values of the a's, the b's remaining the same. But the following diagrams show that the first arrangement is possible, and the second is not. The third diagram is an additional illustration of the method. The rows are arranged in succession beginning with the top.

	6	5	4	2	1
5	B	B	B	B	B
4	B	B	B	B	W
3	B	B	B	W	W
3	B	B	B	W	W
2	B	B	W	W	W
1	B	W	W	W	W

	6	5	4	2	1
5	B	B	B	B	B
4	B	B	B	B	W
4	B	B	B		
3					
1					
1					

	2	2	2	2	1
3	W	B	B	B	W
3	B	W	W	B	B
2	W	B	B	W	W
1	B	W	W	W	W

I first deduced the conditions of inequality from the preceding method, but the following proof of the results obtained seems preferable.

Having given a set of a's and a set of b's satisfying equations (1), (2), (3), (4), we can construct a rectangle having the given a's and also one having the given b's. Let v be the actual weight of any ρ of the rows of the first rectangle, w the hypothetical weight of any ρ of the rows of the second rectangle on the supposition that as many black squares as possible are forced into the ρ rows. Then if the two rectangles can be identical, that is, if the a's and b's can coexist, we must have

$$v \leqq w \text{ for all values of } \rho \text{ from 0 to } n \text{ (incl.)} \qquad \text{......................(5)};$$

that is to say, if there be a_k rows containing k black squares each, &c.,

$$\Sigma a_k . k \leqq b_1 + 2b_2 + \dots + \dots + (\rho - 1) b_{\rho-1} + \rho(b_\rho + b_{\rho+1} + \dots + b_n) \text{ ......(6)},$$

$$\text{where } \rho = \Sigma a$$

This set of conditions which is necessary is also sufficient; for if we take any one row containing m_1 squares out of ρ of the given rows, the condition for being able to place it is

$m_1 \leqq b_1 + b_2 + \dots + b_n$, a relation which is, by hypothesis, satisfied as a particular case of (5).

It will be seen that the actual weight of the $(\rho-1)$ remaining rows of the given rows must be $v-m_1$, and the hypothetical weight of the remaining $(\rho-1)$ rows out of the given columns must be $w-m'_1$.

And $v-m_1 \overline{\underline{<}}\, w-m_1$, since $v \overline{\underline{<}}\, w$. So for any $(\rho-1)$ rows.

Hence the reduced sets of a's and b's satisfy the corresponding inequality to (5); they also obviously satisfy the reduced equations. Therefore another row can be placed, and so on.

To apply (5) we see that for a given value of ρ, w is constant and v is a maximum when the rows are those of greatest weight. Therefore suppose that the rows are taken from those of greatest weight.

Again, if we denote $v_\rho - v_{\rho-1}$ by δv, and $w_\rho - w_{\rho-1}$ by δw, we need only consider those values of ρ in which $\delta v - \delta w$ is positive followed by a negative or zero for the value $(\rho+1)$. Now δv is constant, so long as the final row falls amongst rows of the same weight, and equals weight of each one of those rows; also $\delta w = b_\rho + b_{\rho+1} + \ldots + b_n$ [see 6] is positive, and decreases as ρ increases. Therefore $\delta v - \delta w$ can only change from positive to negative or zero as the final row passes from rows of one weight to the next lower weight.

Suppose the values of the a's which are greater than zero are

$$a_\mu,\ a_{\mu'} \ldots\ldots a_{\mu_r} \ldots\ldots;\quad \mu,\ \mu',\ \ldots\ldots\ \mu_r \ldots\ldots \text{ being in descending order.}$$

Then from (6) it is sufficient to show that

$$\mu a_\mu + \mu' a_{\mu'} + \ldots + \mu_r a_{\mu_r} \overline{\underline{<}}\, b_1 + 2b_2 + \ldots + (\rho-1)\, b_{\rho-1} + \rho\,(b_\rho + \ldots b_n) \ldots\ldots (7),$$

where $a_\mu + a_{\mu'} + \ldots a_{\mu_r} = \rho$, for all cases where at the same time

$$\mu_r > b_\rho + b_{\rho+1} + \ldots b_n \text{ and } \mu_{r+1} < b_{\rho+1} + \ldots\ldots + b_n \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

supposing that $v \overline{\underline{<}}\, w$ also for the extreme cases when $\rho = 1$ and when $\rho = n$. But when $\rho = n$, $v = w = p$; and when $\rho = 1$ there is a very simple test by inspection to ascertain whether (5) is satisfied, for we have

$$\mu \overline{\underline{<}}\, b_1 + b_2 + \ldots\ldots + b_n \overline{\underline{<}}\, m - b_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

If $b_0 = 0$, (9) is satisfied; but if b_0 be > 0, at least b_0 of the a's, from a_m to a_{m-b_0+1} inclusive, must be equal to zero. From symmetry the like relations must hold for a digit greater than 0 at either extremity of either of the numbers representing the a's and b's.

Again, subtracting (7) from the respective values of p in (2) and (4), writing σ for $n-\rho$, and inserting the zero values of the a's, we have the simpler form

$$\tau a_\tau + \ldots\ldots + 1 \,.\, a_1 + 0 \,.\, a_0 \overline{\underline{>}}\, \sigma b_n + \ldots\ldots + 1 \,.\, b_{n-\sigma+1} + 0 \,.\, b_{n-\sigma} \ldots\ldots\ldots (10),$$

where $\sigma = a_\tau + a_{\tau-1} + \ldots\ldots + a_0$; also τ may have any value between μ_{r+1} and $\mu_r - 1$ inclusive, since the corresponding a's, except $a_{\mu_{r+1}}$, are zero; therefore from (8) we may take

$$\tau = b_n + b_{n-1} + \ldots\ldots\ldots + b_{n-\sigma},$$

Hence taking in the symmetrical relations to (10), the following set of conditions, in addition to the test by inspection given by (9), is sufficient for the possibility of arranging in a rectangle two sets of a's and b's, satisfying equations (1), (2), (3), (4).

Cut off from the opposite ends of the two sets, all pairs of partial sets, having the property that the sum of the digits in each partial set is one less than the number of digits in the other.

Take any such pair, and count the weight of each set either in ascending or descending, but in the same order; then the weight of the set whose zero-weight digit is an extreme digit of one of the original sets, must be greater than the weight of the other.

Thus, in the previous example of the failure of construction

$$[1,\ 2,\ 1,\ 0,]\ 2,\ 0 \text{ and } 1,\ 1,\ [1,\ 0,\ 1,\ 1,\ 0],$$

$$1\,.\,4+1\,.\,2+1\,.\,1 \text{ should be } \overline{\overline{>}} \text{ but is } < 1\,.\,3+2\,.\,2+1\,.\,1,$$

or $$0\,.\,3+1\,.\,2+2\,.\,1+1\,.\,0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 0\,.\,4+1\,.\,3+1\,.\,2+1\,.\,0.$$

3088. (Proposed by C. W. MERRIFIELD, F.R.S.)—The condition that the shortest distances between the opposite edges of a tetrahedron shall meet in a point, is either that the bisectors of the edges shall be at right angles to one another, or else that

$$p^2(-\sin^2\alpha+\sin^2\beta+\sin^2\gamma) = q^2(\sin^2\alpha-\sin^2\beta+\sin^2\gamma)$$
$$= r^2(\sin^2\alpha+\sin^2\beta-\sin^2\gamma),$$

where p, q, r are the bisectors, and α, β, γ the angles which they make with one another.

Solution by the PROPOSER.

Refer the tetrahedron to the bisectors of opposite edges as axes, and therefore to its centre of gravity as origin.

Call these bisectors $2p, 2q, 2r$; the angles between them a, b, c; and the angles between the coordinate planes A, B, C. Now a, b, c, A, B, C will be the sides and angles of a spherical triangle. There will be *two* tetrahedra, whose edges are the diagonals of faces of the same parallelepiped. We shall choose the one whose four points are

$$(+p,\ +q,\ +r),\ (+p,\ -q,\ -r),\ (-p,\ +q,\ -r),\ (-p,\ -q,\ +r).$$

The equations of the two edges parallel to the plane of (xy) will be

$$\frac{x}{p}=\frac{y}{q} \text{ and } z=r \ldots\ldots\ (1); \quad \frac{x}{p}=\frac{y}{-q} \text{ and } z=-r \ldots\ldots\ (2).$$

The next thing we have to do is to find the equation of a line perpendicular to the plane of xy. To do this, I shall make the distance from a fixed point on the axis of z to a variable point on the plane of xy a minimum.

The (distance)^2 between two points in oblique coordinates is

$$(x_1-x)^2+(y_1-y)^2+(z_1-z)^2+2(y_1-y)(z_1-z)\cos a + \ldots\ldots$$

Let one point be $(0, 0, z_1)$, and the other $(x, y, 0)$; then the (distance)^2 becomes $\quad x^2+y^2+z_1^2-2yz_1 \cos a-2z_1x \cos b+2xy \cos c$,

which is to be made a minimum, x and y being *two* independent variables, and all the other quantities constant.

Equating to zero the partial differential coefficients, we get

$$x+y \cos c-z_1 \cos b = 0, \quad x \cos c+y-z_1 \cos a = 0,$$

which give $\quad \dfrac{x}{\cos b-\cos a \cos c} = \dfrac{y}{\cos a-\cos b \cos c} = \dfrac{z_1}{\sin^2 c}.$

A line parallel to this through the origin will be

$$\frac{x}{\cos a \cos c-\cos b} = \frac{y}{\cos b \cos c-\cos a} = \frac{z}{\sin^2 c} \quad \text{...............} \quad (3),$$

and this line will be perpendicular to the plane of xy.

We have now to make a line parallel to this perpendicular pass through the two edges whose equations are (1) and (2); that is to say, that the equation of a line parallel to (3) must be satisfied identically by some point which fulfils the equation (1), and so for (2).

(1) and (3) give $\quad \dfrac{x-pk}{\cos a \cos c-\cos b} = \dfrac{y-qk}{\cos b \cos c-\cos a} = \dfrac{z-r}{\sin^2 c} \quad \text{..........} \quad (4),$

(2) and (3) give $\quad \dfrac{x-pl}{\cos a \cos c-\cos b} = \dfrac{y+ql}{\cos b \cos c-\cos a} = \dfrac{z+r}{\sin^2 c} \quad \text{..........} \quad (5),$

in which the indeterminate quantities k and l are to have such values as will make (4) and (5) represent the same line, for which it is sufficient to eliminate k and l.

By subtraction, $\quad \dfrac{p(k-l)}{\cos a \cos c-\cos b} = \dfrac{q(k+l)}{\cos b \cos c-\cos a} = \dfrac{2r}{\sin^2 c} \quad \text{..........} \quad (6);$

by addition, $\quad \dfrac{2x-p(k+l)}{\cos a \cos c-\cos b} = \dfrac{2y-q(k-l)}{\cos b \cos c-\cos a} = \dfrac{2z}{\sin^2 c} \quad \text{..........} \quad (7).$

The values of $(k-l)$ and $(k+l)$ from (6), being substituted in (7), give

$$\frac{qx \sin^2 c-pr(\cos b \cos c-\cos a)}{q(\cos a \cos c-\cos b)} = \frac{py \sin^2 c-qr(\cos a \cos c-\cos b)}{p(\cos b \cos c-\cos a)} = z,$$

which may be written as

$$\frac{qx \sin c+pr \sin b \cos A}{q \sin a \cos B} = \frac{py \sin c+qr \sin a \cos B}{p \sin b \cos A} = -z,$$

by introducing the ordinary formulæ

$$\cos A = \frac{\cos a-\cos b \cos c}{\sin b \sin c}, \quad \text{\&c.},$$

of spherical trigonometry. This is the equation of one of the shortest distances.

The equations of the three shortest distances are thus:—

$$-x = \frac{ry \sin a+qp \sin c \cos B}{r \sin b \cos C} = \frac{qz \sin a+rp \sin b \cos C}{q \sin c \cos B} \quad \text{.......} \quad (8),$$

$$\frac{rx \sin b+pq \sin c \cos A}{r \sin a \cos C} = -y = \frac{pz \sin b+rq \sin a \cos C}{p \sin c \cos A} \quad \text{.......} \quad (9),$$

$$\frac{qx \sin c+pr \sin b \cos A}{q \sin a \cos B} = \frac{py \sin c+qr \sin a \cos B}{p \sin b \cos A} = -z \quad \text{......} \quad (10).$$

For convenience of putting down the determinant, I write (8) and (9) in the form

$$\frac{x-0}{-1}=\frac{y+\dfrac{pq}{r}\dfrac{\sin c\cos B}{\sin a}}{\dfrac{\sin b\cos C}{\sin a}}=\frac{z+\dfrac{rp}{q}\dfrac{\sin b\cos C}{\sin a}}{\dfrac{\sin c\cos B}{\sin a}},$$

$$\frac{x+\dfrac{pq}{r}\dfrac{\sin c\cos A}{\sin b}}{\dfrac{\sin a\cos C}{\sin b}}=\frac{y-0}{-1}=\frac{z+\dfrac{rq}{p}\dfrac{\sin a\cos C}{\sin b}}{\dfrac{\sin c\cos A}{\sin b}};$$

and the condition that two of these should meet is

$$\begin{vmatrix} -1\ , & \dfrac{\sin a\cos C}{\sin b}, & -\dfrac{pq}{r}\dfrac{\sin c\cos A}{\sin b} \\ \dfrac{\sin b\cos C}{\sin a}, & -1\ , & \dfrac{pq}{r}\dfrac{\sin c\cos B}{\sin a}\ . \\ \dfrac{\sin c\cos B}{\sin a}, & \dfrac{\sin c\cos A}{\sin b}, & \dfrac{rp}{q}\dfrac{\sin b\cos C}{\sin a}-\dfrac{rq}{p}\dfrac{\sin a\cos C}{\sin b} \end{vmatrix}=0,$$

which, on expansion and reduction, becomes

$$\left\{-q^2r^2\sin^2 a\sin^2 C+r^2p^2\sin^2 b\sin^2 C+p^2q^2\sin^2 c\,(\sin^2 B-\sin^2 A)\right\}\cos C=0,$$

and this is the condition that two of the shortest distances should meet in a point.

Now, by spherical trigonometry, $\dfrac{\sin a}{\sin A}=\dfrac{\sin b}{\sin B}=\dfrac{\sin c}{\sin C}$.

Introducing these relations, $\sin^2 C$ divides out; and dividing also by $p^2q^2r^2$, we get, for the three conditions that the shortest distances should meet two and two,

$$\left(\frac{\sin^2 C-\sin^2 B}{p^2}-\frac{\sin^2 B}{q^2}+\frac{\sin^2 C}{r^2}\right)\cos A=0\ \ldots\ldots\ldots\ldots(11),$$

$$\left(\frac{\sin^2 A}{p^2}+\frac{\sin^2 A-\sin^2 C}{q^2}-\frac{\sin^2 C}{r^2}\right)\cos B=0\ \ldots\ldots\ldots\ldots(12),$$

$$\left(-\frac{\sin^2 A}{p^2}+\frac{\sin^2 B}{q^2}+\frac{\sin^2 B-\sin^2 A}{r^2}\right)\cos C=0\ \ldots\ldots\ldots\ldots(13).$$

These are satisfied independently of the values of p, q, r, by $\cos A=0$, $\cos B=0$, $\cos C=0$; that is to say, by the dihedral angles between the coordinate planes becoming severally rectangular. All three will pass through the same point if the coordinates be rectangular.

Supposing, however, that $\cos A$, $\cos B$, $\cos C$ do not vanish, we may divide them out. The determinant of the resulting equations vanishes, for we then have identically

$$(11)\sin^2 A+(12)\sin^2 B+(13)\sin^2 C=0.$$

If, on the other hand, we take any two of them separately, we find, on reduction,

$$p^2(-\sin^2 A+\sin^2 B+\sin^2 C)=q^2(\sin^2 A-\sin^2 B+\sin^2 C)$$
$$=r^2(\sin^2 A+\sin^2 B-\sin^2 C),$$

or $$\frac{\sin^2 A}{p^2(q^2+r^2)} = \frac{\sin^2 B}{q^2(p^2+r^2)} = \frac{\sin^2 C}{r^2(p^2+q^2)},$$

(in either of which A, B, C may be replaced by a, b, c,) as the condition that the three shortest distances should meet in a point, unless the coordinates be rectangular, when the condition is also fulfilled. In that case p, q, r are indeterminate, and the edges of the tetrahedron are diagonals of faces of a rectangular parallelepiped.

The simplest case of the general form is given by $p=q=r$, $\sin A = \sin B = \sin C$. The tetrahedron is then a right pyramid on an equilateral base.

3031. (Proposed by G. M. MINCHIN, B.A.)—Show that the modulus derived from K by a transformation of the order $2n$ (n being any odd integer) is given by the equation

$$\mu = \left\{\frac{(1-\Delta am\,\omega)(1-\Delta am\,3\omega)\ldots\{1-\Delta am\,(n-2)\,\omega\}}{(1+\Delta am\,\omega)(1+\Delta am\,3\omega)\ldots\{1+\Delta am\,(n-2)\,\omega\}}\right\}^2,$$

μ being the derived modulus, and $\omega = \dfrac{K}{n}$, K being, as usual, the complete elliptic function of the first kind.

Solution by the PROPOSER.

Let λ be the modulus derived from K by Jacobi's first transformation for the order n. Then we know that the modulus μ, derived from λ by Lagrange's transformation, $= \dfrac{1-\lambda'}{1+\lambda'}$, λ' being the complement of λ.

Now if ψ is the amplitude derived from ϕ for the order n, we can easily see at once the equation

$$1-\Delta(\psi) = (1-\Delta amu)\{1-\Delta am\,(u+2\omega)\}\ldots[1-\Delta am\,\{u+2(n-1)\,\omega\}],$$

ω being (as used by Jacobi) $= \dfrac{K}{n}$. And similarly,

$$1+\Delta(\psi) = (1+\Delta amu)\{1+\Delta am\,(u+2\omega)\}\ldots[1+\Delta am\,\{u+2(n-1)\,\omega\}].$$

Now we know that when $u=\omega$, $\psi=\frac{1}{2}\pi$, and therefore $\Delta\psi = (1-\lambda^2)^{\frac{1}{2}} = \lambda'$;

hence $$\frac{1-\lambda'}{1+\lambda'} = \frac{(1-\Delta am\,\omega)(1-\Delta am\,3\omega)\ldots\{1-\Delta am\,(2n-1)\,\omega\}}{(1+\Delta am\,\omega)(1+\Delta am\,3\omega)\ldots\{1+\Delta am\,(2n-1)\,\omega\}}.$$

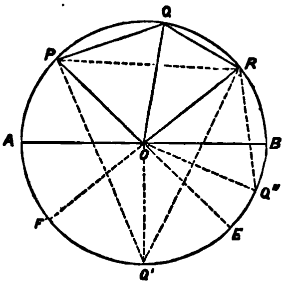
But evidently
$$\Delta am\,(2n-1)\,\omega = \Delta am\,\omega,$$
$$\Delta am\,(2n-3)\,\omega = \Delta am\,3\omega, \text{ \&c.};$$
$$\ldots\ldots \qquad \ldots\ldots$$

therefore, finally,

$$\mu = \frac{1-\lambda'}{1+\lambda'} = \left\{\frac{(1+\Delta am\,\omega)(1-\Delta am\,3\omega)\ldots\{1-\Delta am\,(n-2)\,\omega\}}{(1+\Delta am\,\omega)(1+\Delta am\,3\omega)\ldots\{1+\Delta am\,(n-2)\,\omega\}}\right\}^2.$$

2547. (Proposed by STEPHEN WATSON.)—Find the average area of the quadrilaterals whose angular points are the centre of a given circle and three points taken at random on its circumference.

Solution by the PROPOSER.



First, let the three points P, Q, R be all above the diameter AB, and put AO (O being the centre) $= a$, $\angle$ AOP $= \alpha$, POR $= \beta$, POQ $= \gamma$; then the area of the quadrilateral OPQR is

$$\tfrac{1}{2}a^2\left\{\sin\gamma+\sin(\beta-\gamma)\right\}.$$

Doubling because P, Q, R may all lie below AB, and multiplying by 6 because P, Q, R may be interchanged in 6 ways, we have for the sum of the areas of the quadrilaterals, when P, Q, R lie all above or all below AB,

$$6a^2\int_0^\pi a\,d\alpha\int_0^{\pi-\alpha} a\,d\beta\int_0^\beta a\,d\gamma\left\{\sin\gamma+\sin(\beta-\gamma)\right\} = 6a^5(\pi^2-4)\ \ldots\ (1).$$

Produce now PO and RO to meet the circle again in E and F, and let Q be on the arc EF, as at Q′. Put $\angle$ EOQ′ $= \gamma'$; then in this case the quadrilateral may be composed of any two of the triangles OPR, OPQ′, ORQ′; hence the sum of the areas is twice the triangle PRQ′, that is,

$$a^2\left\{\sin\beta+\sin\gamma'+\sin(\beta-\gamma')\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

Doubling because two of the points may lie below AB and one above in the same manner, and multiplying by 6 as before, we have, for the sum of the areas when the quadrilaterals are re-entrant,

$$12\int_0^\pi a\,d\alpha\int_0^{\pi-\alpha} a\,d\beta\int_0^\beta a\,d\gamma'\,(2) = 12a^5\pi^2\ \ldots\ldots\ldots\ldots\ldots\ (3).$$

Again, let Q be on the arc BE, as at Q″, and put EOQ″ $= \gamma''$; then the area of the quadrilateral OPRQ″ is

$$\tfrac{1}{2}a^2\left\{\sin\beta+\sin(\beta+\gamma'')\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4);$$

and doubling because Q may lie on the arc AF, also doubling again and multiplying by 6 for the same reasons as before, we have for the sum of the areas in this case,

$$24\int_0^\pi a\,d\alpha\int_0^{\pi-\alpha} a\,d\beta\int_0^\alpha a\,d\gamma''\,(4) = 6a^5(\pi^2+4)\ \ldots\ldots\ldots\ldots\ (5).$$

For the number of extra quadrilaterals in the second case above, we have

$$24\int_0^\pi a\,d\alpha\int_0^{\pi-\alpha} a\,d\beta\int_0^\beta a\,d\gamma' = 4a^3\pi^3,$$

and this, added to $8a^3\pi^3$, the number of ways the three points can be taken on the circumference, gives $12a^3\pi^3$ for the whole number of quadrilaterals, and therefore the required average is

$$\frac{(1)+(3)+(5)}{12a^3\pi^3} = \frac{2a^2}{\pi} = \frac{2}{\pi^2} \text{ of the given circle.}$$

3136. (Proposed by the Rev. E. Hill.)—Inside a cylindrical bucket, partly filled with water, floats another. Enough water is poured into the outer one to fill it. Had the same quantity been poured into the inner one, the outer one would still have been filled. If the descent of the inner bucket in the second case be equal to its rise in the first, show that the ratio of the radii of the buckets is $\sqrt{2}$.

Solution by James Dale.

Let r_1, r_2 be the radii of the outer and inner buckets respectively, h the depth of the surface of the water in the outer bucket; then to fill it requires a quantity of water $= \pi\,(r_1^2 - r_2^2)\,h$. But when the inner bucket is depressed through the depth h, it displaces a quantity of water $= \pi r_2 h$. And by the question this is also sufficient to fill the outer bucket;

therefore $\qquad \pi\,(r_1^2 - r_2^2)\,h = \pi\, r_2^2\, h,$

therefore $\qquad r_1 = r_2\sqrt{2}.$

3155. (Proposed by R. Tucker, M.A.)—If the angular escribed radii of a spherical triangle be r_1, r_2, r_3, and if $r_1+r_2+r_3=\pi$, show that

$$2 \tan R \ \tan s \ \tan S = \cos s;$$

and if, at the same time, $S \doteq \pi$, then

$$R_1 + R_2 + R_3 = \tfrac{1}{2}\pi.$$

Solution by the Proposer.

Referring throughout to Todhunter's *Spherical Trigonometry*, we have, see p. 61 (4) and p. 60 (4),

$$\lambda = \tan r_1 + \tan r_2 + \tan r_3 - \tan r = \frac{4N \sin S}{\sin A \sin B \sin C},$$

$$\mu = \tan r_1 \ \tan r_2 \ \tan r_3 \ \tan r = \frac{4N^4}{\sin^2 A \sin^2 B \sin^2 C},$$

therefore $\qquad \dfrac{\lambda^2}{\mu} = \dfrac{4\sin^2 S}{N^2};$

and as $$r_1 + r_2 + r_3 = \pi \quad \text{..................................} \quad (\alpha),$$
we have, by p. 66, Ex. 1,
$$\lambda^2 = \tan^2 r \cos^4 s, \quad \mu = \tan^2 r \sin^2 s;$$
therefore $$\frac{4\sin^2 S}{N^2} = \frac{\cos^4 s}{\sin^2 s}, \quad \text{or} \quad -\frac{\cos^2 s}{\sin s} = 2\frac{\sin S}{N} = -2\tan R \tan S;$$
therefore $$2\tan R \tan s \tan S = \cos s, \text{ see p. 63 (2).}$$

Again, bearing in mind (α), we have, see p. 61 (1),
$$\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C \sin^3 s = (\tan \tfrac{1}{2}A + \tan \tfrac{1}{2}B + \tan \tfrac{1}{2}C) \sin s;$$
hence, if $S = \pi$, then $$\sin^2 s = 1 \quad \text{or} \quad s = \tfrac{1}{2}\pi \quad \text{............................} \quad (\beta).$$

Now $\tan R_1 \tan R_2 + \tan R_2 \tan R_3 + \tan R_3 \tan R_1$, p. 64 (1),
$$= \frac{\tan \frac{1}{2}a \tan \frac{1}{2}b + \tan \frac{1}{2}b \tan \frac{1}{2}c + \tan \frac{1}{2}c \tan \frac{1}{2}a}{\cos^2 S} = 1, \text{ by } (\beta);$$
therefore $$R_1 + R_2 + R_3 = \tfrac{1}{2}\pi.$$

3040. (Proposed by A. Martin.)—Required a solution of the biquadratic equation $$4x^4 + 4ax^3 + 4bx + ab = 0.$$

Solution by Henry Hoskins; R. Tucker, M.A.; *the* Proposer; *and many others.*

We have $$4x^4 + 4ax^3 = -4bx - ab;$$
therefore $$(2x^2 + ax + z)^2 = (4z + a^2)x^2 + 2(az - 2b)x + (z^2 - ab);$$
and the right-hand side will be a complete square if
$$(az - 2b)^2 = (4z + a^2)(z^2 - ab), \quad \text{or} \quad 4z^3 = b(4b + a^3).$$
And z being now known, the square can be completed, and the four roots of the equation found.

2936. (Proposed by the Rev. J. Wolstenholme, M.A.)—A heavy particle is projected so as to move on a rough inclined plane whose angle of inclination is equal to the angle of friction; prove that the hodograph of its path is a parabola, and the intrinsic equation of the path is
$$\frac{ds}{d\phi} = \frac{u^2}{4g \sin \alpha} \left\{ \frac{3\sin\phi + 2\sin^2\phi}{(1+\sin\phi)^2} + \log \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right) \right\},$$
u being the velocity at the highest point, and $\tan \alpha$ the coefficient of friction.

Solution by the PROPOSER.

The acceleration of the particle is compounded of two equal (and constant) accelerations, one along the tangent and the other in a fixed direction; hence, in the hodograph, the velocity will be compounded of two equal velocities, one along the radius vector and the other in a fixed direction; the tangent to the hodograph will therefore make equal angles with the radius vector and a fixed straight line; or the hodograph is a parabola, the focus being pole.

The equations of motion of the particle are

$$\frac{d^2s}{dt^2} = -f + f\sin\phi, \quad \left(\frac{ds}{dt}\right)^2 = f\rho\cos\phi = f\frac{ds}{d\phi}\cos\phi,$$

ϕ being the angle which the tangent makes with the horizon, ρ the radius of curvature, and $f = g\sin\alpha$;

therefore $\dfrac{d^2s}{dt^2}\dfrac{dt}{ds} + \dfrac{1-\sin\phi}{\cos\phi}\dfrac{d\phi}{dt} = 0$; or $\dfrac{ds}{dt}(1+\sin\phi) = u$;

and the intrinsic equation is $\dfrac{ds}{d\phi} = \dfrac{u^2}{f(1+\sin\phi)^2\cos\phi}$;

or, integrating, and measuring the arc from the highest point,

$$s = \frac{u^2}{4f}\left\{\frac{3\sin\phi + 2\sin^2\phi}{(1+\sin\phi)^2} + \log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right)\right\}.$$

If x be the horizontal space described from the highest point,

$$\frac{dx}{d\phi} = \frac{ds}{d\phi}\cos\phi = \frac{u^2}{f(1+\sin\phi)^2},$$

and the whole horizontal space described is

$$\frac{u^2}{f}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{(1+\sin\phi)^2} \equiv \frac{u^2}{f}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{(1+\cos\phi)^2} = \frac{2u^2}{3f}.$$

The particle will then ultimately run directly down the plane with constant velocity $\frac{1}{2}u$, but the time required is $\dfrac{u}{f}\displaystyle\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\cos\phi\,(1+\sin\phi)}$, or is infinite. The straight line $x = \dfrac{2u^2}{3f}$ is then an asymptote to the path.

3158. (Proposed by Dr. JAMES MATTESON.)—It is required to find n^3 consecutive numbers of the natural series, the sum of whose cubes shall be a cube.

Solution by ASHER B. EVANS, M.A.

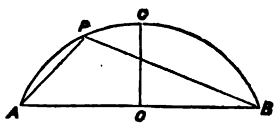
The sum of the cubes of n^3 consecutive numbers of the natural series is

$$n^6\left\{\tfrac{1}{2}(n^3+1)\right\}^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

It is evident that (1) cannot be a cube unless $\frac{1}{2}(n^3+1)$ is a cube. But Euler has shown (see Euler's *Algebra*, Part II., Art. 247) that $x^3 \pm y^3 = 2z^3$ is impossible except in the evident case of $y = x$; therefore $\frac{1}{2}(n^3+1)$ cannot equal a cube except when $n = 1$.

2991. (Proposed by STEPHEN WATSON.)—Through the extremities of the major axis of an ellipse two lines are drawn in random directions: find the chance of their intersecting within the ellipse.

Solution by the PROPOSER.



Let A, B be the extremities of the major diameter of the ellipse, and OC the semi-minor; also let P be any point in the arc AC, and join AP, BP. Put $AO = a$, $OC = b$, $\angle PAB = \phi$, $PBA = \theta$, and $m = \dfrac{b}{a}$; then

$$\tan\phi \tan\theta = m^2, \quad \text{or} \quad \theta = \tan^{-1}(m^2\cot\phi).$$

When $\tan\phi < m$, the favourable cases intersecting to the *left* of OC are

$$\int_0^{\tan^{-1}m} d\phi \int_0^{\phi} d\theta = \tfrac{1}{2}(\tan^{-1}m)^2.$$

When $\tan\phi > m$, the favourable cases intersecting in the same way are

$$\int_{\tan^{-1}m}^{\frac{1}{2}\pi} \tan^{-1}(m^2\cot\phi)\, d\phi.$$

Hence doubling because the lines may intersect to the *right* of OC, and dividing by $\frac{1}{4}\pi^2$, the number of ways two lines can be drawn through A and B to meet above AB, the required chance is

$$p = \frac{2}{\pi^2}\left\{(\tan^{-1}m)^2 + 2\int_{\tan^{-1}m}^{\frac{1}{2}\pi} \tan^{-1}(m^2\cot\phi)\, d\phi\right\} \text{............(1).}$$

Expanding $\tan^{-1}(m^2\cot\phi)$, and remembering that

$$\cot^n\phi\, d\phi = \cot^{n-2}\phi\,(\operatorname{cosec}^2\phi - 1)\, d\phi = -\cot^{n-2}\phi\, d(\cot\phi) - \cot^{n-2}\phi\, d\phi,$$

we find without much difficulty

$$2\int_{\tan^{-1}m}^{\frac{1}{2}\pi} \tan^{-1}(m^2\cot\phi)\, d\phi = m^2(1 + \tfrac{1}{3}m^4 + \tfrac{1}{5}m^8 + \&c.) \log\frac{1+m^2}{m^2}$$

$$- \tfrac{1}{2}m^4 + \tfrac{1}{3}m^6(\tfrac{1}{2} - m^2) - \tfrac{1}{4}m^8(\tfrac{1}{3} - \tfrac{1}{2}m^2 + m^4) + \&c.$$

But $$m^2(1 + \tfrac{1}{3}m^4 + \tfrac{1}{5}m^8 + \&c.) = \tfrac{1}{2}\log\frac{1+m^2}{1-m^2};$$

hence (1) becomes

$$p = \frac{2}{\pi^2}\left\{(\tan^{-1} m)^2 + \tfrac{1}{2}\log\frac{1+m^2}{1-m^2}\,\log\frac{1+m^2}{m^2} - \tfrac{1}{3}m^4 + \tfrac{1}{5}m^6\left(\tfrac{1}{2}-m^2\right)\right.$$
$$\left. - \tfrac{1}{7}m^8\left(\tfrac{1}{3}-\tfrac{1}{2}m^2+m^4\right) + \tfrac{1}{9}m^{10}\left(\tfrac{1}{4}-\tfrac{1}{3}m^2+\tfrac{1}{2}m^4-m^6\right) - \&c.\right\}.$$

For $m = \frac{3}{4}$, this becomes $p = \cdot 1927$; and as m varies from $\frac{3}{4}$ to 1, p varies from $\cdot 1927$ to $\cdot 25$. For less values of m than $\frac{3}{4}$, the above series converges rapidly.

2957. (Proposed by G. A. OGILVIE.)—ABC′ and A′B′C are two co-polar triangles. Let the straight line formed by joining the points of intersection of CB and C′A′ and of C′B and AC be denoted by ED. Let O be the intersection of CB and C′B′, then the straight line formed by joining O to the intersection of ED and CC′ is the fourth harmonic to CB, C′B′ and the axis of the triangles.

Solution by W. H. LAVERTY, B.A.

The property is a projective one. Project the figure, then, so that the axis of the triangles becomes the line at infinity. Then O is at infinity. Call the intersection of DE and CC′, F. Then plainly DE and CC′ are the diagonals of a parallelogram, and therefore bisect one another in F; therefore the fourth harmonic to C, F, C′, is a point at infinity, and therefore the fourth harmonic ray of the pencil OC, OF, OC′, is the line at infinity; that is, the axis of the triangles.

3153. (Proposed by the Rev. A. F. TORRY, M.A.)—A conic circumscribes a quadrilateral ABCD; E is the pole of AB, F that of CD; and AB, CD intersect in O. Through E is drawn a straight line, cutting the conic in Q, R, and CD in M; and upon this line a point P is taken to make the range (QPRM) harmonic. Show that the locus of P will be a conic AEBCFD, the tangents to which at E, F pass through O; and that the tangents from O to the first conic will pass through the four points in which the common tangents to the two conics touch the second.

I. *Solution by* W. H. LAVERTY, B.A.

Project the conic into a circle, and AB to infinity; so that A and B become the circular points, and E becomes the centre of a circle

$$x^2 + y^2 = a^2.$$

Let the equation to CD be $x = c$; therefore the locus of P is a circle (therefore passing through A and B) whose equation is

$$c(x^2+y^2) = a^2 x,$$

which evidently passes through C, D, E, F.

The tangents at E and F will pass through O, which is the point at infinity on CD. Also the common tangents to the two circles will easily be shown to touch the new locus in points whose abscissæ are $\pm a$, which are the points through which the tangents from O to the old circle pass. These being all position-properties will be equally true in the case of the conic.

II. *Solution by* T. COTTERILL, M.A.

Let the given conic be called (C). Then the locus of P is the *quadric inverse* of the line CD to the point E and the conic (C), according to the method explained by Dr. Hirst. It follows from this, and is easily seen, that the locus is a conic (K) through the six points (A, B, C, D, E, F). Also, since E and F are the poles of AB and CD meeting in O to the conic (C), O is the pole EF to the same conic, and therefore to the conic (K) which passes through the points A, B, C, D. Hence, P being a point on (K), OP and the tangent to (K) at P cut AB harmonically. Also the locus of points at which the conic (C) and the distance AB is seen under harmonic angles is a conic, which is found to be (K) by observing that the pairs of tangents from the points (A, B, C, D, E, F) to the conic (C) cut the distance AB harmonically. We have then two pairs of lines at each point of (K) which cut AB harmonically; and taking the point at an intersection of a tangent to (C) from O, the last part of the question directly follows.

One example of such a pair of conics, (C) and (K), is a central conic and the concentral circle, at the points of which the conic is seen under a right angle; and another is two circles, the centre of one lying on the circumference of the other.

3087. (Proposed by the Rev. R. TOWNSEND, M.A., F.R.S.)—If A, B, C be the three spherical centres of any three coaxal circles on the surface of a sphere, a, b, c their three spherical radii, and α, β, γ their three angles of intersection with any arbitrary circle of the sphere; prove the following general relation,

$$\sin BC \sin a \cos \alpha + \sin CA \sin b \cos \beta + \sin AB \sin c \cos \gamma = 0.$$

I. *Solution by the* PROPOSER.

For, denoting by P and r the centre and radius of the arbitrary circle, since, by the fundamental formula of Spherical Trigonometry,

$$\cos PA = \cos a \cos r + \sin a \sin r \cos \alpha,$$
$$\cos PB = \cos b \cos r + \sin b \sin r \cos \beta,$$
$$\cos PC = \cos c \cos r + \sin c \sin r \cos \gamma;$$

since also, by a well known general property,

$$\sin BC . \cos PA + \sin CA . \cos PB + \sin AB . \cos PC = 0;$$

and since, finally, from the coaxality of the three circles,

$$\sin BC . \cos a + \sin CA . \cos b + \sin AB . \cos c = 0;$$

therefore, &c.

NOTE.—From the above general relation, all the properties of coaxal

circles in a plane, established from the analogous relation for the plane, in Art. 193 of my *Modern Geometry*, are seen directly to be true for coaxal circles on a sphere also. That from the former they follow indirectly for the latter case, is of course also evident by inversion from any arbitrary point in space.

II. *Solution by* J. J. WALKER, M.A.

Let O be the centre of the arbitrary circle, ρ its radius, Q one of the points in which it meets the circle ABC, and OS a perpendicular arc on the same circle, while P is that point on ABC from which tangents to the three given circles are equal. Then, if

$$k = \frac{\cos a}{\cos \mathrm{PA}} = \frac{\cos b}{\cos \mathrm{PB}} = \frac{\cos c}{\cos \mathrm{PC}},$$

$$\begin{aligned}\sin\rho \sin a \cos\alpha &= \cos \mathrm{OA} - \cos\rho \cos a \\ &= \cos \mathrm{OS} \cos \mathrm{SA} - k \cos \mathrm{OS} \cos \mathrm{SQ} \cos \mathrm{PA} \\ &= \cos \mathrm{OS} \left\{ \cos(\mathrm{PA}-\mathrm{PS}) - k \cos \mathrm{SQ} \cos \mathrm{PA} \right\} \\ &= \cos \mathrm{OS} \left\{ (\cos \mathrm{PS} - k \cos \mathrm{SQ}) \cos \mathrm{PA} + \sin \mathrm{PS} \sin \mathrm{PA} \right\}.\end{aligned}$$

Hence it easily appears that

$$\sin \mathrm{BC} \sin a \cos\alpha + \sin \mathrm{CA} \sin b \cos\beta + \sin \mathrm{AB} \sin c \cos\gamma$$

is of the form

$$p\,(\sin \mathrm{BC} \cos \mathrm{PA} + \sin \mathrm{CA} \cos \mathrm{PB} + \sin \mathrm{AB} \cos \mathrm{PC}) + q\,(\sin \mathrm{BC} \sin \mathrm{PA} + \ldots).$$

Now we have

$$\begin{aligned}2\sin \mathrm{BC} \cos \mathrm{PA} &= \sin(\mathrm{BC}+\mathrm{PA}) + \sin(\mathrm{BC}-\mathrm{PA}), \quad \text{and} \quad \mathrm{BC} = \mathrm{PC}-\mathrm{PB}; \\ 2\sin \mathrm{CA} \cos \mathrm{PB} &= \sin(\mathrm{CA}+\mathrm{PB}) + \sin(\mathrm{CA}-\mathrm{PB}), \quad \text{and} \quad \mathrm{CA} = \mathrm{PA}-\mathrm{PC}; \\ 2\sin \mathrm{AB} \cos \mathrm{PC} &= \sin(\mathrm{AB}+\mathrm{PC}) + \sin(\mathrm{AB}-\mathrm{PC}), \quad \text{and} \quad \mathrm{AB} = \mathrm{PB}-\mathrm{PA}.\end{aligned}$$

From these relations it at once follows that the coefficient of p above vanishes identically; and it may similarly be shown that the coefficient of q also vanishes identically: therefore &c.

By following analogous steps the relation *in plano*, viz.,

$$\mathrm{BC} \,.\, a \,.\, \cos\alpha + \mathrm{CA} \,.\, b \,.\, \cos\beta + \mathrm{AB} \,.\, c \,.\, \cos\gamma = 0,$$

may be proved for three coaxal circles having centres A, B, C, radii a, b, c, and cut by any arbitrary circle at angles α, β, γ respectively.

3050. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—If a, b, c be the three sides of the triangle determined on the surface of a sphere by the centres of any three small circles of the sphere; A, B, C its three angles; p, q, r the spherical radii of the three circles; and k that of the circle orthogonal to the three: prove that

$$\sec^2 k = \frac{l^2 + m^2 + n^2 - 2mn\cos \mathrm{A} - 2nl \cos \mathrm{B} - 2lm \cos \mathrm{C}}{4 \sin s \sin(s-a) \sin(s-b) \sin(s-c)};$$

where $l = \cos p \sin a$, $m = \cos q \sin b$, $n = \cos r \sin c$.

I. *Solution by* J. J. Walker, M.A.

Let O be the centre of the orthogonal circle, and OE, OF perpendicular arcs from O on AC, AB respectively; then, if AE $=\beta$, AF $=\gamma$, OA $=\delta$, the formula $\tan^2\delta=\dfrac{\tan^2\beta+\tan^2\gamma-2\tan\beta\tan\gamma\cos A}{\sin^2 A}$ holds (see questions); and since $\cos$ AE : $\cos$ CE $=\cos p:\cos r$, it easily follows that $\tan\beta=\dfrac{\cos r-\cos p\cos b}{\cos p\sin b}$. Similarly $\tan\gamma=\dfrac{\cos q-\cos p\cos c}{\cos p\sin c}$.

Now $\sec^2 k=\cos^2 p\,(1+\tan^2\delta)$, whence

$$\begin{aligned}\sin^2 b\sin^2 c\sin^2 A\sec^2 k^2 &= \cos^2 p\sin^2 b\sin^2 c\sin^2 A\\ &\quad+(\cos q-\cos p\cos c)^2\sin^2 b+(\cos r-\cos p\cos b)^2\sin^2 c\\ &\quad-2(\cos q-\cos p\cos c)(\cos r-\cos p\cos b)\sin b\sin c\cos A.\end{aligned}$$

The coefficient of $\cos^2 p$ in this expression is

$$\sin^2 b\sin^2 c\sin^2 A+\sin^2 b\cos^2 c+\sin^2 c\cos^2 b-2\sin b\sin c\cos b\cos c\cos A,$$

which $=\sin^2 a$. The coefficient of $2\cos p\cos q$ is

$$-\sin^2 b\cos c+\sin b\sin c\cos b\cos A;$$

which $=-\sin a\sin b\cos C$. Similarly, the coefficient of $2\cos p\cos r$ is equal to $-\sin a\sin c\cos B$. The coefficients of $\cos^2 q$, $\cos^2 p$, and $2\cos q\cos r$, are, identically, $\sin^2 b$, $\sin^2 c$, and $-\sin b\sin c\cos A$ respectively. Hence $\sec^2 k=$ &c. &c.

It is readily found that, if arc OE $=e$, the position of O relatively to the sides of the triangle ABC is determined by

$$\cos^2 e=\frac{\cos^2 p+\cos^2 r-2\cos p\cos r\cos b}{\sin^2 b}$$

and two corresponding equations.

An analogous method to that adopted in the above solution leads, on the plane, to the formula

$$k^2=\frac{a^2b^2c^2+\Sigma a^2(p^2-q^2)(p^2-r^2)-\Sigma a^2(b^2+c^2-a^2)p^2}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$$

for the square of radius of circle cutting orthogonally three circles whose radii are p, q, r and centres A, B, C respectively; and if $2\sigma^2=a^2+b^2+c^2$, the distance of the radical centre of the three circles from the side BC is equal to

$$\frac{(\sigma^2-a^2-p^2)a^2+(\sigma^2-b^2)r^2+(\sigma^2-c^2)q^2}{2abc\sin A}.$$

II. *Solution by the* Proposer.

In the known formula connecting the cosines of the three sides a, b, c of the triangle with those of the three connectors d, e, f of its opposite vertices, with the centre of the orthogonal circle, or with any other point on the sphere, viz.,

$$\begin{aligned}&(\cos^2 a+\cos^2 b+\cos^2 c)+(\cos^2 d+\cos^2 e+\cos^2 f)\\ &\quad-(\cos^2 a\cos^2 d+\cos^2 b\cos^2 e+\cos^2 c\cos^2 f)\\ &+2(\cos b\cos c\cos e\cos f+\cos c\cos a\cos f\cos d+\cos a\cos b\cos d\cos e)\\ &-2(\cos a\cos b\cos c+\cos a\cos e\cos f+\cos b\cos f\cos d+\cos c\cos d\cos e)=1\end{aligned}$$

(see Salmon's *Geometry of Three Dimensions*, 2nd Edit., Art. 52), substituting for the three latter cosines their values, viz.,

$$\cos d = \cos k \cos p, \quad \cos e = \cos k \cos q, \quad \cos f = \cos k \cos v,$$

and for the coefficient of $\sec^2 k$ in the result, viz.,

$$1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c,$$

its known equivalent $4 \sin s \sin (s-a) \sin (s-b) \sin (s-c)$, the above is the immediate result.

NOTE.—In the particular case when $p=0$, $q=0$, $r=0$, in which case k is, of course, the radius of the circle which circumscribes the triangle abc, the above gives at once, as it ought, the known value for $\tan k$ in terms of the sides, viz.,

$$\tan^2 k = \frac{(1-\cos a)(1-\cos b)(1-\cos c)}{2 \sin s \sin (s-a) \sin (s-b) \sin (s-c)}$$
$$= \frac{4 \sin^2 \tfrac{1}{2}a \sin^2 \tfrac{1}{2}b \sin^2 \tfrac{1}{2}c}{2 \sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

3096. (Proposed by R. W. GENESE.)—For the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$,

$$\frac{\text{the product of the four normals from } (x, y)}{\text{the product of the two tangents from } (x, y)} = \frac{b^2x^2 + a^2y^2}{a^2 - b^2}.$$

Hence deduce Mr. Wolstenholme's elegant theorem:—"The product of the three normals to a parabola, divided by the product of the two tangents, is constant, and equal to one-fourth of the latus rectum."

I. *Solution by* J. J. WALKER, M.A.

Let (x', y') be one of the four points on the ellipse, the normals at which meet in (x, y), r the intercept between (x', y') and (x, y), N that between (x', y') and the transverse axis. Then

$$r^2 = \frac{a^4 (x'-x)^2 \mathrm{N}^2}{b^4 x'^2} = \frac{a^2 (x'-x)^2 (a^2 - e^2 x'^2)}{b^2 x'^2} \;\ldots\ldots\ldots\ldots\ldots (1).$$

To form the equation which gives the four values of x', we have to eliminate y' between

$$b^2x'^2 + a^2y'^2 - a^2b^2 = 0 \;\ldots\ldots (2), \quad \text{and} \quad b^2yx' - a^2xy' + c^2x'y' = 0 \;\ldots\ldots (3).$$

The result is,

$$c^4x'^4 - 2a^2c^2xx'^3 + a^2(a^2x^2 + b^2y^2 - c^4)x'^2 + 2a^4c^2xx' - a^6x^2 = 0 \;\ldots\ldots\ldots (4),$$

let the roots of which be $x_1 \ldots x_4$, corresponding to the four values $r_1 \ldots r_4$ of r. Then, if $z = a - ex'$, or $x' = -\dfrac{az}{c} + \dfrac{a^2}{c}$, the product $z_1z_2z_3z_4$ will be got by forming the leading and final terms of the substitution of this value of x' in (4); viz., $a^4z^4 \ldots + \dfrac{a^6b^2}{c^2}(c^2 + x^2 + y^2 + 2cx) = 0.$

Similarly, if $z' = a + ex'$, the equation for z' is formed from that in z by simply changing the sign of c. Hence

$$(a^2 - e^2x_1^2) \dots (a^2 - e^2x_4^2) = \frac{a^4b^4}{c^4}\left\{c^4 - 2c^2(x^2 - y^2) + (x^2 + y^2)^2\right\} = \frac{a^4b^4}{c^4}\rho^2\rho'^2 \quad \dots\dots\dots\dots (5),$$

if $\rho\rho'$ are the lines joining (x, y) with the foci. Also, from (4),

$$x_1 \dots x_4 = \frac{a^4x^2}{c^4} \quad \dots\dots\dots\dots (6).$$

To form the extreme terms of equation whose roots are $\xi = x' - x$ from (4), the leading term is $c^4\xi^2$, and the final term the result of substituting x for x', *i.e.* $b^2k^4x^2$, where $k^4 = b^2x^2 + a^2y^2 - a^2b^2$. Hence

$$\xi_1\xi_2\xi_3\xi_4 = \frac{b^4k^4x^2}{c^4} \quad \dots\dots\dots\dots (7).$$

From (1), (5), (6), (7), $r_1r_2r_3r_4 = \frac{k^4\rho\rho'}{c^2}$. But (see Quest. 3099) if t_1, t_2 are the tangents drawn from (x, y), $t_1t_2 = \frac{k^4\rho\rho'}{b^2x^2 + a^2y^2}$, since k^4, $b^2x^2 + a^2y^2$ are in the same ratio as the lines drawn from (x, y) to the middle point of the chord of contact and to the centre of the ellipse; therefore &c.

Now if the origin be transferred to the vertex, $\frac{b^2x^2 + a^2y^2}{a^2 - b^2}$ becomes, after division by a^2,

$$\frac{y^2 + \frac{2m}{a}x^2 - 4mx + 2ma}{1 - \frac{2m}{a}},$$

where $4m$ is the latus-rectum, or $\frac{2b^2}{a}$. In the limit, when a becomes infinite, the fourth normal becomes equal to $2a$. Dividing the preceding fraction, therefore, again by $2a$, and proceeding to the limit, $\frac{r_1r_2r_3}{t_1t_2} = m$.

II. *Solution by the* Rev. J. Wolstenholme, M.A.

If θ be the eccentric angle of the foot of one of the normals drawn from (x, y) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the length of the normal is

$$\left\{(x - a\cos\theta)^2 + (y - b\sin\theta)^2\right\}^{\frac{1}{2}},$$

where θ is given by the equation $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \equiv c^2 \quad \dots\dots\dots (1)$;

whence the product of the four normals is

$$(x - a\cos\theta_1)(x - a\cos\theta_2)(x - a\cos\theta_3)(x - a\cos\theta_4)$$
$$\times\left[\left(1 + \frac{a^2}{b^2}\tan^2\theta_1\right)\left(1 + \frac{a^2}{b^2}\tan^2\theta_2\right)\left(1 + \frac{a^2}{b^2}\tan^2\theta_3\right)\left(1 + \frac{a^2}{b^2}\tan^2\theta_4\right)\right]^{\frac{1}{2}},$$

where $\theta_1, \theta_2, \theta_3, \theta_4$ are the four roots of (1).

Now (1) may be written

$(by\cos\theta)^2-(c^2\cos\theta-ax)^2(1-\cos^2\theta)=0\equiv c^4(\cos\theta-\cos\theta_1)(\cos\theta-\cos\theta_2)\ldots,$

whence $c^4(x-a\cos\theta_1)(x-a\cos\theta_2)(x-a\cos\theta_3)(x-a\cos\theta_4)$

$$=a^2b^2x^2y^2-x^2(c^2-a^2)^2(a^2-x^2)\equiv b^2x^2(a^2y^2+b^2x^2-a^2b^2).$$

So also (1) may be written

$$(ax\tan\theta-by)^2(1+\tan^2\theta)-c^4\tan^2\theta=0$$

$$\equiv a^2x^2(\tan\theta-\tan\theta_1)(\tan\theta-\tan\theta_2)(\tan\theta-\tan\theta_3)(\tan\theta-\tan\theta_4);$$

hence, putting, as usual, i for $\sqrt{-1}$, we have

$$(ibx-by)^2\left(1-\frac{b^2}{a^2}\right)+c^4\frac{b^2}{a^2}$$

$$\equiv a^2x^2\left(i\frac{b}{a}-\tan\theta_1\right)\left(i\frac{b}{a}-\tan\theta_2\right)\left(i\frac{b}{a}-\tan\theta_3\right)\left(i\frac{b}{a}-\tan\theta_4\right),$$

$$(ibx+by)^2\left(1-\frac{b^2}{a^2}\right)+c^4\frac{b^2}{a^2}=a^2x^2\left(i\frac{b}{a}+\tan\theta_1\right)\ \ldots\ldots\ldots;$$

therefore $$\left(\frac{b^2c^2}{a^2}(y^2-x^2+c^2)\right)^2+4\frac{b^4c^4}{a^4}x^2y^2$$

$$=a^4x^4\left(\frac{b^2}{a^2}+\tan^2\theta_1\right)\left(\frac{b^2}{a^2}+\tan^2\theta_2\right)\left(\frac{b^2}{a^2}+\tan^2\theta_3\right)\left(\frac{b^2}{a^2}+\tan^2\theta_4\right),$$

or $$(x^2-y^2-a^2+b^2)^2+4x^2y^2$$

$$=\frac{b^4x^4}{c^4}\left(1+\frac{a^2}{b^2}\tan^2\theta_1\right)\left(1+\frac{a^2}{b^2}\tan^2\theta_2\right)\left(1+\frac{a^2}{b^2}\tan^2\theta_3\right)\left(1+\frac{a^2}{b^2}\tan^2\theta_4\right),$$

or the product of the four normals drawn from x, y is

$$\frac{a^2y^2+b^2x^2-a^2b^2}{a^2-b^2}\left\{(x^2-y^2-a^2+b^2)^2+4x^2y^2\right\}^{\frac{1}{2}}.$$

So if α, β be the eccentric angles of the points of contact of the two tangents, their product is

$$\tan^2\tfrac{1}{2}(\alpha-\beta)(a^2\sin^2\alpha+b^2\cos^2\alpha)^{\frac{1}{2}}(a^2\sin^2\beta+b^2\cos^2\beta)^{\frac{1}{2}}.$$

Now $\quad\tan^2\tfrac{1}{2}(\alpha-\beta)=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1,\quad \cos\alpha\cos\beta=\dfrac{1-\dfrac{y^2}{b^2}}{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}},$

and $\quad 1+\tan^2\theta-\left(\dfrac{x}{a}+\dfrac{y}{b}\tan\theta\right)^2\equiv\left(1-\dfrac{y^2}{b^2}\right)(\tan\theta-\tan\alpha)(\tan\theta-\tan\beta);$

therefore $\quad 1-\dfrac{b^2}{a^2}-\left(\dfrac{x}{a}+i\dfrac{y}{b}\right)^2=\left(1-\dfrac{y^2}{b^2}\right)\left(i\dfrac{b}{a}-\tan\alpha\right)\left(i\dfrac{b}{a}-\tan\beta\right);$

$\therefore\quad \dfrac{(x^2-y^2-a^2+b^2)^2+4x^2y^2}{a^4}=\left(1-\dfrac{y^2}{b^2}\right)^2\left(\dfrac{b^2}{a^2}+\tan^2\alpha\right)\left(\dfrac{b^2}{a^2}+\tan^2\beta\right),$

or the product of the tangents is

$$\frac{\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1\right)}{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}}\left\{(x^2-y^2-a^2+b^2)^2+4x^2y^2\right\}^{\frac{1}{2}},$$

or $$\frac{\text{product of the normals}}{\text{product of the tangents}} = \frac{b^2x^2+a^2y^2}{a^2-b^2}$$

= product of the perpendiculars from (x, y) on the asymptotes.

This extension of a problem set by me in Christ's College was communicated to me by Mr. Ferrers immediately afterwards. The case of the parabola is immediately deduced, as the limit of

$$\frac{b^2(x-a)^2+y^2}{(a^2-b^2)\left\{(2a-x)^2+y^2\right\}}, \text{ when } a, b \text{ become } \infty, \text{ and } \frac{b^2}{a} = c.$$

3128. (Proposed by M. W. Crofton, F.R.S.)—Prove that, X being any function of x, $f'(D)X = \left\{f(D)x - xf(D)\right\}X$,

also that $$\phi\left(\frac{d}{dD}\right)f(D)X = \phi(x_2-x_1)f(D)X,$$

where x_1 stands for x *preceding* $f(D)$, x_2 for x *following* $f(D)$; the meaning being that, if $\phi(x_2-x_1)$ is expressed by expansion or otherwise in a series of terms such as $u_{x_1}v_{x_2}$, the value of the symbolical expression above will consist of a corresponding series of terms such as $u_x f(D) v_x X$.

As an instance, $$e^{h\frac{d}{dD}}f(D)X = e^{-hx}f(D)e^{hx}X.$$

I. *Solution by* Samuel Roberts, M.A.

As to the first part of the question, it is proved by Leibnitz's theorem, which gives $$f(D)xX = xf(D)X + f'(D)X.$$
As to the more general form, let $\pi, \rho, \rho_1, \rho_2 \ldots \rho_n$ be symbols of operation such that with respect to the subject we have

$$\rho_1 = \rho\pi - \pi\rho, \quad \rho_2 = \rho_1\pi - \pi\rho_1, \quad \ldots\ldots \quad \rho_n = \rho_{n-1}\pi - \pi\rho_{n-1}.$$

Then we have $\rho_2 = (\rho\pi - \pi\rho)\pi - \pi(\rho\pi - \pi\rho) = \rho\pi^2 - 2\pi\rho\pi + \pi^2\rho$,

$$\rho_3 = \rho_1\pi^2 - 2\pi\rho_1\pi + \pi^2\rho_1 = \rho\pi^3 - 3\pi\rho\pi^2 + 3\pi^2\rho\pi - \pi^3\rho;$$

and, generally, if $\rho_{m-1} = \rho\pi^{m-1} - (m-1)\pi\rho\pi^{m-2} + \ldots$, we have

$$\rho_m = \rho_{m-1}\pi - \pi\rho_{m-1} = \rho\pi^m - m\pi\rho\pi^{m-1} + \&c.$$

For the coefficient of the argument $\pi^p\rho\pi^{m-p}$ is evidently

$$\pm\left\{\frac{(m-1)\ldots(m-p)}{1\ldots p} + \frac{(m-1)\ldots(m-p+1)}{1\ldots p-1}\right\} = \pm\frac{m(m-1)\ldots(m-p+1)}{1\ldots p}.$$

But we have the system

$$f'(D) = f(D)x - xfD, \quad f''(D) = f'(D)x - xf'D, \quad \ldots\ldots\ldots$$
$$f^n(D) = f^{n-1}(D)x - xf^{n-1}(D);$$

and for $\left(\frac{d}{dD}\right)^n$ the theorem of the question follows directly, since π may represent x, and ρ, ρ_1, &c. may represent $f(D), f'(D)$, &c. Hence also, by

immediate inference, the theorem is true for a rational and integral function, say $\phi\left(\frac{d}{d\mathrm{D}}\right)$.

II. *Solution by* J. J. WALKER, M.A.

1. Let $f(\mathrm{D}) = \Sigma a_r \mathrm{D}^r$, then, by Leibnitz's Theorem,

$$\mathrm{D}^r x\mathrm{X} = x\mathrm{D}^r\mathrm{X} + r\mathrm{D}^{r-1}\mathrm{X}, \text{ or } \frac{d}{d\mathrm{D}}(a_r\mathrm{D}^r)\,\mathrm{X} = a_r\mathrm{D}^r x\mathrm{X} - xa_r\mathrm{D}^r\mathrm{X},$$

whence $$f'(\mathrm{D})\,\mathrm{X} = f(\mathrm{D})\,x\mathrm{X} - xf(\mathrm{D})\,\mathrm{X}.$$

2. Suppose $$\phi\left(\frac{d}{d\mathrm{D}}\right) = \Sigma b_r \left(\frac{d}{d\mathrm{D}}\right)^r,$$

and let it be true for some one given (positive integral) value of r that

$$(3)\quad \left(\frac{d}{d\mathrm{D}}\right)^r f(\mathrm{D})\,\mathrm{X} = (x_2 - x_1)^r f(\mathrm{D})\,\mathrm{X}$$

$$= f(\mathrm{D})\,x^r\mathrm{X} - rxf(\mathrm{D})\,x^{r-1}\mathrm{X} + \frac{r.r-1}{1.2}x^2 f(\mathrm{D})\,x^{r-2}\mathrm{X}\ldots,$$

then $$\left(\frac{d}{d\mathrm{D}}\right)^{r+1} f(\mathrm{D})\,x$$

$$= \frac{d}{d\mathrm{D}}f(\mathrm{D})\,x^r\mathrm{X} - rx\frac{d}{d\mathrm{D}}f(\mathrm{D})\,x^{r-1}\mathrm{X} + \frac{r.r-1}{1.2}x^2\frac{d}{d\mathrm{D}}f(\mathrm{D})\,x^{r-2}\mathrm{X}\ldots\ldots$$

$$= (\text{by } 1)\ \ f(\mathrm{D})\,x^{r+1}\mathrm{X} - xf(\mathrm{D})\,x^r\mathrm{X} - rxf(\mathrm{D})\,x^r\mathrm{X} + rx^2 f(\mathrm{D})\,x^{r-1}\mathrm{X}$$

$$+ \frac{r.r-1}{1.2}x^2 f(\mathrm{D})\,x^{r-1}\mathrm{X} - \frac{r.r-1}{1.2}x^3 f(\mathrm{D})\,x^{r-2}\mathrm{X} - \ldots\ldots + \ldots\ldots,$$

that is, $$\left(\frac{d}{d\mathrm{D}}\right)^{r+1} f(\mathrm{D})\,\mathrm{X}$$

$$= f(\mathrm{D})\,x^{r+1}\mathrm{X} - (r+1)\,xf(\mathrm{D})\,x^r\mathrm{X} + \frac{r+1.r}{1.2}x^2 f(\mathrm{D})\,x^{r-1}\mathrm{X} - \ldots\ldots$$

$$= (x_2 - x_1)_{r+1} f(\mathrm{D})\,\mathrm{X}.$$

But (3) is true when $r=1$, by (1); therefore it is true for $r=2$, $r=3$, or generally for every positive integral value of r; therefore

$$\Sigma b_r \left(\frac{d}{d\mathrm{D}}\right)^r f(\mathrm{D})\mathrm{X} = \Sigma b_r\,(x_2 - x_1)^r f(\mathrm{D})\,\mathrm{X}.$$

3033. (Proposed by R. W. GENESE.)—If from the three angles of a triangle circumscribed about a conic straight lines be drawn parallel to those joining the middle points of the opposite sides to the centre of the conic; they will cointersect.

I. *Solution by the* Rev. R. TOWNSEND, M.A., F.R.S.

If A, B, C be the three vertices of any triangle, and A′, B′, C′ the three middle points of its respectively opposite sides; then, the two triangles

ABC and A′B′C′ being similar, and oppositely placed with respect to each other, when three lines through A, B, C concur to any common point P, the three parallels to them through A′, B′, C′ concur to the homologous point P′, which connects with P by a line PP′ passing through their centre of similitude O and there cut internally in their ratio of similitude 2 : 1. And, for the same reason precisely, if A, B, C, D be the four vertices of any tetrahedron, and A′, B′, C′, D′ the four centres of gravity of its respectively opposite faces; then, the two tetrahedra ABCD and A′B′C′D′ being similar, and oppositely placed with respect to each other, when four lines through A, B, C, D concur to any common point P, the four parallels to them through A′, B′, C′, D′ concur to the homologous point P′, which connects with P by a line PP′ passing through their centre of similitude O and there cut internally in their ratio of similitude 3 : 1.

II. *Solution by* J. Dale; R. Tucker, M.A.; *and the* Proposer.

Let the equation to the conic in areal coordinates be

$$\sqrt{lx}+\sqrt{my}+\sqrt{nz}=0,$$

the coordinates of the centre (O) are given by

$$\frac{x}{m+n}=\frac{y}{n+l}=\frac{z}{l+m},$$

and the coordinates of the middle point (D) of BC are

$$\frac{x}{0}=\frac{y}{1}=\frac{z}{1};$$

hence the equation to OD is $(m-n)x+(m+n)(y-z)=0$,
and this line meets the line at infinity in the point

$$\frac{x}{-(m+n)}=\frac{y}{m}=\frac{z}{n};$$

therefore the equation to the parallel through A is $\frac{y}{m}=\frac{z}{n}$, and the equations of the corresponding lines through B and C are $\frac{z}{n}=\frac{x}{l}$, $\frac{x}{l}=\frac{y}{m}$; therefore the three lines meet in the point $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

[The solution of this question is included in the solution to Quest. 2996.]

An easy way of remembering the Formulæ of Reduction *in the Integral Calculus, both as regards their Investigation and their Application. By* H. M'Coll.

Commit to memory the accompanying tables of exponents, which may be done in a minute or two if learnt ***vertically*** and ***not horizontally***.

Tables of Exponents.					
I.			II.		
1	$m+n$	$p-1$	1	m	p
2	$m-n$	$p+1$	2	$m-n$	$p+1$
3	$m+n$	p	3	m	$p+1$
4	$m-n$	p	4	$m-n$	$p+1$
5	m	$p-1$	5	m	p
6	m	$p+1$	6	m	$p+1$

Let $$\int x^m (a+bx^n)^p\,dx,$$
(which, for shortness' sake, we will denote throughout by W,) be the given integral to be reduced; and, in each formula, let V denote the integral upon which W is made to depend.

The explanation of Table I. is this:

In the 1st formula, $V = \int x^{m+n} (a+bx^n)^{p-1}\,dx.$

&c. &c. &c.

In the 6th formula, $V = \int x^m (a+bx^n)^{p+1}\,dx.$

The meaning of Table II. will appear from the following six independent investigations of the six formulæ, to remember which is the object of the two Tables:

1. In the first formula, we take (see Table II., 1) the integral $\int x^m (a+bx^n)^p\,dx$ (which we have already denoted by W); and integrating this by parts, we get $W = A - CV$,
in which A denotes the first term (or algebraical part) on the right-hand side of the equation, and C denotes the constant outside the integral V. Thus the integral W is made to depend upon the integral V.

2. In the second formula, we take (see Table II., 2) the integral $\int x^{m-n} (a+bx^n)^{p+1}\,dx$ (which, we have said, is V for the second formula); and integrating this by parts, we get $V = A - CW$,
in which, as in the preceding case, A denotes the first term on the right side of the equation, and C denotes the constant outside the integral W.

Transposing, we get $$W = \frac{A-V}{C},$$
thus making the integral W depend upon the integral V.

3. In the third formula, we take (see Table II., 3) the integral $\int x^m (a+bx^n)^{p+1}\,dx$ (which we will denote by U); and integrating this by parts, we get $U = A - CV$,
in which A and C are to be interpreted as in the preceding cases. Also, by the usual process of separation, we get

$$U = \int x^m (a+bx^n)^p (a+bx^n)\,dx$$
$$= a\int x^m (a+bx^n)^p\,dx + b\int x^{m+n} (a+bx^n)^p\,dx = aW + bV.$$

Equating the two values of U, we get

$$W = \frac{A-(b+C)V}{a},$$

thus, as before, making the integral W depend upon the integral V.

4. In this formula, we take (see II., 4) the integral $\int x^{m} \quad (a+bx^{n})^{p+1}dx$ (which we will denote by U); and integrating this by parts, we get

$$U = A - CW,$$

in which A and C are to be interpreted as in the preceding cases. Separating also, as before, we get

$$\begin{aligned} U &= \int x^{m-n}(a+bx^{n})^{p}(a+bx^{n})\,dx \\ &= a\int x^{m-n}(a+bx^{n})^{p}\,dx + b\int x^{m}(a+bx^{n})^{p}\,dx = aV + bW. \end{aligned}$$

Equating the two values of U, we get

$$W = \frac{A-aV}{b+C}.$$

5. In this formula, we take (see II., 5) the integral $\int x \quad (a+bx^{n})^{p}\,dx$ (which we have already denoted by W); and integrating this by parts, we get

$$W = A - CU,$$

in which U denotes the integral which appears on the right-hand side of the equation, and A and C are to be interpreted as in the preceding cases. We get also, as before, by separation,

$$\begin{aligned} W &= \int x^{m}(a+bx^{n})^{p-1}(a+bx^{n})\,dx \\ &= a\int x^{m}(a+bx^{n})^{p-1}\,dx + b\int x^{m+n}(a+bx^{n})^{p-1}\,dx = aV + bU. \end{aligned}$$

Equating the two values of W, we get $U = \dfrac{A-aV}{b+C}$;

and substituting this value of U in either of the values of W, we get

$$W = \frac{aCV+bA}{b+C}.$$

6. In this formula, we take (see II., 6) the integral $\int x^{m}(a+bx^{n})^{p+1}dx$ (which is the value of V in this formula); and integrating by parts, we get

$$V = A - CU,$$

in which A, C, and U are to be interpreted as before. We also get, as before, by separation,

$$\begin{aligned} V &= \int x^{m}(a+bx^{n})^{p}(a+bx^{n})\,dx \\ &= a\int x^{m}(a+bx^{n})^{p}\,dx + b\int x^{m+n}(a+bx^{n})^{p}\,dx = aW + bU. \end{aligned}$$

Equating the two values of U obtained from these equations, we get

$$W = \frac{(b+C)V - bA}{aC}.$$

Thus we see that, as Table I. informs us which of the six formulæ it will be necessary or advantageous for us to use, so Table II. informs us what function we must first operate upon in order to obtain this formula or apply the method of its investigation to any example. The six formulæ are in the same order of succession as in Todhunter's *Integral Calculus;* but from the point of view of this article I have found it convenient to denote the exponent of x, outside the bracket, in W, by m instead of (as is usually done) by $m-1$.

Magic Squares for 1870. *By* Artemas Martin.

Let n be the number of cells in a side of a magic square, x the first term of the series of numbers in arithmetical progression composing it, and d their common difference. Then n^2 is the number of terms in the series, and the number of cells in the square; $x + (n^2 - 1)\,d$ is the last term, and $\frac{1}{2}n^2\left\{2x + (n^2-1)\,d\right\}$ is the sum of the series. The sum of the numbers in each column or row must be one-nth of the sum of the whole series, therefore

$$\tfrac{1}{2}n\left\{2x + (n^2-1)\,d\right\} = 1870;$$

whence

$$x = \frac{1870}{n} - \frac{(n^2-1)\,d}{2}.$$

d and n may have any integral values that will give x integral and positive.

If $d = 1$,

$$x = \frac{1870}{n} - \frac{n^2-1}{2}.$$

Taking $n=4$, we have $x=460$; taking $n=5$, $x=362$; and taking $n=11$, $x=110$. The squares corresponding to these values are

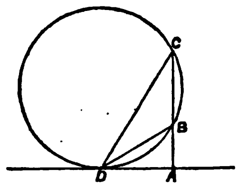
460	475	470	465
472	463	466	469
467	468	473	462
471	464	461	474

362	371	373	379	385
370	372	381	383	364
374	380	382	366	368
378	384	365	367	376
386	363	369	375	377

189	202	215	228	120	122	135	148	161	174	176
177	190	203	216	229	110	123	136	149	162	175
213	226	118	131	133	146	159	172	185	187	200
201	214	227	119	121	134	147	160	173	186	188
128	141	143	156	169	182	195	208	210	223	115
116	129	142	144	157	170	183	196	198	211	224
225	117	130	132	145	158	171	184	197	199	212
152	154	167	180	193	206	219	221	113	126	139
140	153	155	168	181	194	207	209	222	114	127
165	178	191	204	217	230	111	124	137	150	163
164	166	179	192	205	218	220	112	125	138	151

3132. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—A mark is made on a vertical tower at a known height from the horizontal plane; the altitude of this mark and of the top of the tower is observed from a point in the plane: find the probable error of the height of the tower deduced from these observations, in terms of the probable error in the measurement of an angle, and show that the best position of observation is that at which the sum of the two altitudes is a right angle.

Solution by JAMES DALE.



Let AC be the tower, B the mark, D the point of observation: also put

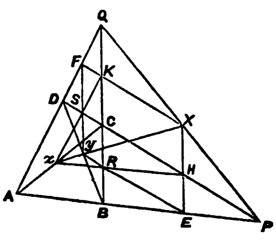
$$AB = a, \quad ADB = \theta_1, \quad ADC = \theta_2,$$
$$\Delta\theta = \text{probable error.}$$

Then $\qquad AC = h = a\dfrac{\tan\theta_2}{\tan\theta_1};$

$$dh = a\left\{\frac{\dfrac{\tan\theta_1}{\cos^2\theta_2} - \dfrac{\tan\theta_2}{\cos^2\theta_1}}{\tan^2\theta_1}\right\}\Delta\theta\; h = \left(\frac{\sin 2\theta_1 - \sin 2\theta_2}{\sin 2\theta_1 \sin 2\theta_2}\right)\Delta\theta.$$

Now $\Delta\theta$ being given, dh is a minimum (1) when $\theta_2 = \theta_1$, and then $h = a$; (2) when $\theta_1 + \theta_2 = 90°$, and in this case D is the point where a circle drawn through BC touches the horizontal; that is, D is the point at which BC subtends a maximum angle.

To prove that the Middle Points of the three Diagonals of a complete Quadrilateral are in one straight line. *By* J. A. MCNEILL.



Let ABCD be one of the component quadrilaterals, and X, Y, Z the middle points of the diagonals of the complete quadrilateral.

Bisect BP, DQ respectively in E and F; and join EX, XF, FY, YE; then H, K, S, R are respectively the middle points of CP, CQ, CD, CB; therefore KS and HR are respectively parallel to AQ and AP.

Now KS being parallel to AQ and drawn through K, the middle point of CQ, must pass through Z; for a similar reason HR passes also through Z. But since RK and SH are parallel to the sides of the parallelogram EF, by a simple exercise (Bland's *Geometrical Problems*, p. 124, Ex. 9, Sect. 4) XY passes through the intersection of KS and HR; but it has been shown that KS and HR intersect in Z, therefore XY passes through Z, that is, X, Y, Z are in one straight line.

[Other proofs of this well-known theorem have been given in previous volumes of the *Reprint*.]

2953. (Proposed by A. MARTIN).—Eliminate y from the equations
$$x^x+y^y=a,\quad x^y+y^x=b.$$

I. *Solution by* R. TUCKER, M.A.

We may write the equations under the form
$$p+q=a,\quad r+s=b,$$
where $\log p=x\log x,\quad \log q=y\log y,$
$$\log r=y\log x,\quad \log s=x\log y,$$
whence $\log p\log q=\log r\log s$ .... (1),
$$\log ps=x\log xy,\quad \log qr=y\log xy;$$
or, $y\log ps=x\log qr,$
and by substitution $\log r=x\log qr\log x\div\log ps$ .... (2).
We have to eliminate q,r,s from the equations (1), (2).

From (2) we get
$$\log r(\log p+\log s-x\log q\log x)=x\log q\log x;$$
or, $\log r(\log p-x\log q\log x)=x\log q\log x-\log p\log q,$
hence $\log r$ is known, and therefore r; and by (1) $\log s$ is known, and therefore s; and we have, since $r+s=b$, a result not containing y.

II. *Solution by the* PROPOSER.

By transposition, $y^y=(a-x^x)$ .... (1).
Taking Napierian logarithms, we have
$y\log y=\log(a-x^x)=z$.......................... (2).
Also, $x^y=b-y^x$, and $y\log x=\log(b-y^x)$ .... (3).
From (2) we have, by the Solution of Quest. 2854, *Reprint* Vol. XIII., p. 40,
$$\log y=1+\tfrac{1}{2}(\log z-1)+\tfrac{1}{16}(\log z-1)^2-\tfrac{1}{192}(\log z-1)^3-\&c.,$$
and
$$y=\frac{z}{1+\frac{1}{2}(\log z-1)+\frac{1}{16}(\log z-1)^2-\frac{1}{192}(\log z-1)^3-\&c.}$$
Substituting the values of y and $\log y$ in (3), we shall have an equation which does not contain y.

ON THE "EXPONENTIAL THEOREM"; BY ARTEMAS MARTIN.

It is proposed to develope a^x into a series of ascending powers of x.

Putting $a^x=1+y$, and taking the Napierian logarithm of each member, we have $(\log a)x=\log(1+y)=y-\frac{1}{2}y^1+\frac{1}{3}y^3-\frac{1}{4}y^4+\&c.;$

whence, by "reversion of series,"

$$y = \frac{(\log a)\,x}{1} + \frac{(\log a)^2 x^2}{1.2} + \frac{(\log a)^3 x^3}{1.2.3} + \frac{(\log a)^4 x^4}{1.2.3.4} + \&c.;$$

$$\therefore\ 1+y = a^x = 1+\frac{(\log a)\,x}{1} + \frac{(\log a)^2 x^2}{1.2} + \frac{(\log a)^3 x^3}{1.2.3} + \frac{(\log a)^4 x^4}{1.2.3.4} + \&c.$$

If $\log a = 1$, then $a = \epsilon$, and

$$\epsilon^x = 1+\frac{x}{1}+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\&c.;$$

and if $x = 1$,

$$\epsilon = 1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\frac{1}{1.2.3.4.5}+\&c.$$

$$= 2{\cdot}718281828459045235360287471352662 49.$$

NOTE ON QUESTION 3167. *By the* Rev. T. P. KIRKMAN, M.A., F.R.S.

This Question was proposed by me in the *Educational Times* for July in the following terms:—

"A turnip is cut at random into a 9-edron, and then thrown into the air. Supposing that it is as likely to come to rest on one face as on another, what is the chance of its settling on a quadrilateral face?"

It is not always a useless thing to ask a clear question which cannot be resolved; but it is silly to print a solution without demonstration, when it is one of which no mortal reader has within his reach either proof or refutation. So the answer to Question 3167 must wait till it is wanted and can make itself understood. It is quite content to wait, with a tall pile of others like it, for another half-century. If we put 8-edron for 9-edron in this question, it is quite proper to give the solution without demonstration, because the numbers and the symmetries of the 8-edra are both known and made known in the *Proceedings of the Royal Society* for January, 1863. Proof of what I have here to say is deducible, though not without trouble, from the Tables there given. The Tables A, which are the important ones, are all correct. In line 7 of Table C, p. 352, for $3^{mo}47 = 1$ read $3^{mo}47 = 2$. In Table D, p. 361, add the edge $(55)^{2ad}16 = 1$ with the zones written in Art. 3 of Table A, p. 360. In Table C, p. 361, line 8, for $Z = 3, \mathbf{2}, \mathbf{0}^2, 0$ read $Z = 1, \mathbf{4}, 0^3$. In the list of epizonal edges, p. 362, add $(53)_{ep}3\mathbf{6} = 3$ in the second zone; and in the last zone, for $(33)_{ep}5\mathbf{6}$ read $(34)_{ep}4\mathbf{6}$. In Table C, p. 367, add the pentagons $5^{mo}57 = 5$, $Z = 2\mathbf{2}0\mathbf{0}$. In Table D, p. 368, line 7, in $(55)_{so}2\mathbf{6} = 1$, put $=2$ for $=1$. In the list of epizonal edges, p. 368, erase all the second line having $Z = \mathbf{420}^2$, and substitute for it the line

$$(46)_{ep}2\mathbf{6} = 2,\quad (44)_{ep}4\mathbf{6} = 2,\quad Z = \cdot, \mathbf{4}, 0^4.$$

In Table C, p. 372, correct the zones of $\mathbf{6}^{di}5\mathbf{7} = 2$ and $\mathbf{4}^{di}7\mathbf{7} = 1$ into $Z = 3\mathbf{20}$. In the first line of asymmetric edges, p. 373, add $(75)_{as}1\mathbf{6} = 7$.

In the second line, for 40 read 36, for 53 read 56; in the third line, for 54 read 55, for 27 read 30, and for 82 read 81; in the fourth line, for 68 read 67, and for 34 read 33.

I regret to have observed these errors too late for correction at the time of printing; but they do not affect my account of the numbers and symmetry of the 8-edra. Yet it is due to the reader, who may try to verify what follows, to refer him to correct tables. He will require also the function $rd_{a_1, a_2} \ldots$ given at the end of my abstract in the *Proceedings of the Royal Society*, for May, 1861.

Two polyedra are different only when no change in the mere lengths or inclinations of their edges can make them identical. If P is the reflected image of P′, but not its own reflected image, P and P′ are both to be enumerated; for the turnip has the same chance of being cut into P′ as into P. We have 423 different 8-edra, which are thus described by their faces: with their various symmetries we have, in this question, nothing to do:—

(33333333) = 2,	(64333333) = 4,	(44333333) = 14,
(53333333) = 2,	(65543333) = 19,	(54433333) = 33,
(73333333) = 1,	(65554333) = 8,	(75443333) = 6,
(55333333) = 5,	(65433333) = 12,	(66443333) = 8,
(55553333) = 4,	(66654333) = 1,	(55544333) = 20,
(66663333) = 1,	(65555433) = 1,	(75544333) = 5,
(66553333) = 2,	(76554333) = 2;	(66544333) = 9,
(55533333) = 7,		(55554433) = 6,
(55555533) = 1,		(75554433) = 1,
(75553333) = 1;		(66554433) = 4,
		(55443333) = 41,
		(74433333) = 2;

(76444333) = 4,	(44443333) = 30,	(65444443) = 2,
(65544433) = 10,	(54444333) = 39,	(64444433) = 4;
(65444333) = 30,	(55444433) = 30,	
(64443333) = 18,	(55554444) = 1,	(44444433) = 8,
(76544433) = 2;	(66444433) = 5,	(54444443) = 4,
	(66544443) = 1,	(55444444) = 1,
	(74444333) = 2,	(66444444) = 1;
	(75444433) = 2,	
	(77444433) = 1,	(44444444) = 1.
	(55544443) = 5;	

This means that there are two 8-edra which have only triangular faces, two which have one pentagon and seven triangles, &c. We see that there are 47 8-edra with each one quadrilateral, 149 with two each, 64 with each three, 116 each having four, 6 having each five, 14 that have each six, and 1 which has eight quadrilaterals. Wherefore

$$\frac{1}{8 \, . \, 423}\left\{47 \, . \, 1 + 149 \, . \, 2 + 64 \, . \, 3 + 116 \, . \, 4 + 6 \, . \, 5 + 14 \, . \, 6 + 1 \, . \, 8\right\} = \frac{1123}{3384},$$

or very nearly $\frac{1}{3}$, is the value of the chance that the 8-edron will settle on a quadrilateral face.

This theory of the polyedra is a very unattractive parvenu, whose history, however, is rather amusing. As it has not been written before, I hope to be excused for reciting it here.

An abstract of my complete *Theory of the Polyedra* appeared in the

Proceedings of the Royal Society, May, 1861, which I circulated with the following note gummed in:—

"The Memoir, of which an abstract is given above, comprises *the whole* of that theory of which the Imperial Institute of France proposed *a part*, in the year 1858, as the subject of their '*Grand Prix de Mathématiques*' for 1861:—'Perfectionner dans quelque point important la Théorie Géométrique des Polyèdres.'

"There are, of course, more ways than one of attacking so extensive a subject; and it may be a matter of a little interest to mathematicians to compare the treatment here briefly sketched of the entire investigation, with the Report of the Referees of the Academy on any Memoirs which may this month be presented to them, if ever they condescend to give one.

"It may easily happen, in a problem of such unusual difficulty, that the Academy may receive no Memoir which they find worthy to be crowned; but after inviting, by a three years' notice, the competition of the men of science of all nations, they will esteem it just and courteous to the Mathematicians who may offer them the homage of their labour, and who deferentially await in silence for nearly a year their award, to give credit to the competitors in some brief form for any new contributions that they may make to science.

"It may be sanctioned by the custom of the Academy, but it is difficult to reconcile such a custom with the renowned politeness of the French people, that the Referees for these prizes, to whom the sought solutions are secrets as profound, and the next inch of progress granite quite as hard, as to the rest of the scientific world, should dismiss in two or three curt lines, without any definite acknowledgment, results confessedly new and important, both in the method and in the matter of the inquiry."

The allusion of the final sentence is to the mysterious disappearance of a second *Grand Prix* (like the other, a gold medal of 3000 francs), which was hung up by the Institute before the jealous sun at the same moment with this Polyedra Medal, and was to be awarded in 1860, the subject of competition for that year being almost as difficult, though not so completely untouched, as that for 1861. Three competitors appeared for the prize of 1860, who were all commended as having done well; but the Academy was not content, and saw no reason either to award her offered medal, or to permit a second competition. The medals of the Institute have very frequently been re-suspended, when not awarded, and when the competitors have done well, and sometimes when they have done ill; but is there another case on record of the withdrawal at once of their Grand Mathematical Prize under similar circumstances? One of the three competitors of 1860 was an English friend of mine—the only one of the three, so far as I can learn, who has since proceeded with that thorny theory. He has thoroughly mastered it, and has published in abstract his completion of it in the *Manchester Proceedings*.

This cooled my courage as a competitor for French honours. I said to myself,—If my friend has fared thus, after presenting, at the invitation of the Imperial Institute, a memoir on a subject about which they knew a little, (yet that little somewhat confusedly, as their Report of 1860 proves,) where is the use of my sending to them a more difficult treatise on a subject of which they know nothing? Will the Academy be content or discontented, if, when she asks me for a small slice, I load her plate with the whole animal? On this I consulted Mr. Punch, and his advice was—Don't. So I waited patiently for the Report of the competition of 1861, which, on its tardy appearance, informed the planet that some eight or nine competitors had accomplished nothing. The medal was this turn re-suspended for a second competition, and in the course of time the inevitable Report in the *Comptes Rendus* announced that the competiters had accomplished twice as much. The poor Academy was now, at the end of four or five years, content upon compulsion. Slowly from her golden box she took one heavy pinch, and then silently committed her two glittering medals of 1858 to the same pocket with her snuff-box. And all persons of

feeling and good breeding religiously abstain from every allusion to the subject henceforth. If she tried so long in vain to make her

"bright eyes
Rain influence and adjudge the prize."

among the small men living, there have since been memorable results put on record from the same quarter in the redistribution of honours among the great men dead. Meanwhile I had presented my entire theory to the Royal Society, who naturally declined either to print or to read a book so dry and useless, thus administering a very proper rebuke to the Academy's *ignotum pro magnifico*. They consigned it to the Archives; and they served me right. If a country clergyman, down in the crowd of the Church's "passing rich," chooses to read lectures to Imperial Institutes, he must even take what comes from Royal Societies. Later, however, they made me, for an Englishman, and a Divine afflicted with science, quite proud and happy, by printing in the *Philosophical Transactions* the two first of my twenty-one sections. This, together with what has appeared in the *Proceedings of the Royal Society*, suffices for my purpose, although, without the demonstrations and tables that should have followed, it is of no use, earthly or celestial, to anybody else. The truths of Nature in our common space of three dimensions may well be left to wait for a century or two, till our eager analysts have discussed the geometries of all the superior dimensions. Yet is it a silly conceit to fancy that the way to the science of molecular forces and combinations may lie, perhaps, through such humble matter of fact as the enumeration and symmetries of polyedra?

3148. (Proposed by J. J. Walker, M.A.)—Referring to Professor Cayley's Question 3126, it may be shown that

$$x : y : z = (a^3+b^3+c^3)(b-c)-(b^3-c^3)\,a : (a^3+b^3+c^3)(c-a)-(c^3-a^3)\,b$$
$$: (a^3+b^3+c^3)(a-b)-(a^3-b^3)\,c$$

is also a point on the curve. What is the geometrical connexion between this and the point $x : y : z = a : b : c$?

Solution by the Proposer.

If $2s = a+b+c$, it may readily be shown that the tangent at (a, b, c) is $(s^2-a^2)\,x+(s^2-b^2)\,y+(s^2-c^2)\,z=0$, which line passes through the point $x : y : z = b^2-c^2 : c^2-a^2 : a^2-b^2$. To find the point where this tangent again meets the curve, eliminating z between their equations, we have

$$\left\{(s^2-c^2)^3-(s^2-a^2)^3+(s^2-c^2)(s^2-a^2)(a^2-c^2)\right\}x^3+(\dots)\,x^2y+(\dots)\,xy^2$$
$$+\left\{(s^2-c^2)^3-(s^2-b^2)^3+(s^2-b^2)(s^2-c^2)(b^2-c^2)\right\}y^3=0 \;\dots\dots\; (1).$$

The coefficient of x^3 reduces identically to $(a^2-c^2)\left\{b^2+2b(a+c)-(a-c)^2\right\}^2$.
Now $a^3+b^3+c^3-(a+b)(b+c)(c+a)=0$ gives

$$(a+c)(a-c)^2 = 2s(a+c)\,b-b^3;$$

and by the aid of this latter equality the coefficient of x^3 above may be thrown into the form $(a-c)\,b^2\left\{(2s)^3-4(a^2+ac+c^2)\,b\right\}$.

Similarly, the coefficient of y^3 in (1) may be thrown into the form

$$(b-c)\,a^2\left\{(2s)^3-4\,(b^2+bc+c^2)\,a\right\}.$$

But (1) must be divisible by $(bx-ay)^2$, since the tangent at (a, b, c) meets the curve twice at that point. The quotient is plainly

$$(a-c)\left\{(2s)^3-4\,(a^2+ac+c^2)\,b\right\}x+(b-c)\left\{(2s)^3-4\,(b^2+bc+c^2)\,a\right\}y,$$

or for the point where the tangent again meets the curve,

$$x:y = (b-c)\left\{(2s)^3-4\,(b^2+bc+c^2)\,a\right\}:(c-a)\left\{(2s)^3-4\,(c^2+ca+a^2)\,b\right\};$$

and, by symmetry,

$$x:z = (b-c)\left\{(2s)^3-4\,(b^2+bc+c^2)\,a\right\}:(a-b)\left\{(2s)^3-4\,(a^2+ab+b^2)\,c\right\}.$$

Finally, since $(a+b+c)^3 \equiv a^3+b^3+c^3+3\,(a+b)\,(b+c)\,(c+a)$,

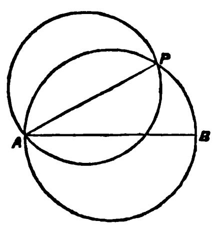
if (a, b, c) be a point on the curve,

$$(a+b+c)^3 = 4\,(a^3+b^3+c^3);$$

therefore the point given by the first three equations in the question is that point on the curve where the tangent at (a, b, c) again meets it.

3174. (Proposed by ARTEMAS MARTIN.)—A radius vector is drawn from a given point in the circumference of a circle, and a circle described on it as diameter: find the average area common to both circles.

Solution by STEPHEN WATSON.



Let A be the given point in the circumference, AB the diameter through A, and AP any chord. Put AB $= 2a$, $\angle$ PAB $= \phi$; then the area common to both circles is

$$\tfrac{1}{2}a^2\pi\cos^2\phi+\tfrac{1}{2}a^2(\pi-2\phi)-a^2\sin\phi\cos\phi\;\ldots\ldots\;(1).$$

An element of the circumference at P is $ad\,(2\phi)$, the number of positions P can take is $2a\pi$, and doubling for the case of P below AB, the required average is

$$\frac{4a}{2a\pi}\int_0^{\frac{1}{2}\pi}(1)\,d\phi=\left(\frac{\pi}{2}-\frac{1}{\pi}\right)a^2.$$

In this solution the points P are supposed to be taken at random on the circumference. The result will be the same if the lines AP are drawn in random directions.

2550. (Proposed by G. O. HANLON.)—Five circles, S′, S″, S_1, S_2, R, are drawn, S′ and S″ being concentric, S_1 and S_2 likewise concentric, and the difference between the radii of S′ and S″ being equal to the difference between those of S_1 and S_2. Then, if two ellipses are drawn, one having for foci the centres of S′ and R, and passing through the intersections of these

circles, the other similarly related to S″ and S_1, the ellipses will intersect in two points on a hyperbola having for foci the centres of the circles R and S_2, and passing through their intersections.

Solution by the PROPOSER.

Denoting the length of a perpendicular from any point to the circumference of the circle S′ by s', and to S″ by s'', &c., and giving to each perpendicular the sign *plus* or *minus* according as it falls on the convex or concave side of a circle, it is easy to show that the equations to the first and second ellipses and the hyperbola are, respectively,

$$s'+r=0,\quad s''+s_1=0,\quad r-s_2=0 \;\ldots\ldots\ldots\ldots\ldots (1,2,3).$$

But by the conditions as to the radii, we have

$$s'-s''=s_1-s_2,\quad \text{or}\quad s''+s_1=s'+s_2.$$

Substituting this value for $s''+s_1$ in (2), and changing signs in (1), we get

$$-s'-r=0,\quad s'+s_2=0,\quad r-s_2=0.$$

But these, on being added together, vanish identically, which proves the theorem.

3032. (Proposed by M. W. CROFTON, F.R.S.)—All circles which touch two given circles are cut orthogonally by the pair of circles which pass through the intersections of the given ones and bisect their angles.

Solution by SAMUEL ROBERTS, M.A.

In Townsend's *Modern Geometry*, Vol. I., p. 259, it is shown that, "if A, B, C be the centres of three coaxal circles; AR, BS, CT their three radii; and α, β, γ their three angles of intersection with any arbitrary circle whose centre is not at infinity, the relation

$$BC.AR\cos\alpha + CA.BS\cos\beta + AB.CT\cos\gamma = 0$$

is true in all cases, whatever the species of the system to which they belong."

In the above expression, put $\alpha=\beta=0$, $\gamma=90°$; then

$$BC.AR + CA.BS = 0.$$

But, having regard to the directions of the lines and comparing triangles, we see that this is the condition for the bisection of the angle of intersection of the first and second circle by the third.

3137. (Proposed by Dr. JAMES MATTESON.)—Find three square numbers in arithmetical progression, such that if from each its root be *subtracted* the three remainders shall be rational squares.

I. *Solution by* Samuel Bills.

Let $\quad l = 2pq-(p^2-q^2), \quad m = p^2+q^2, \quad n = 2pq+(p^2-q^2)$;
then will l^2, m^2, n^2 be three square numbers in arithmetical progression.

Now let lx, mx, nx be the roots of the three squares required in the question; then we must find l^2x^2-lx, m^2x^2-mx, n^2x^2-nx all squares; or, putting $l = a^{-1}$, $m = b^{-1}$, $n = c^{-1}$, we must have

$$x^2-ax = \square, \quad x^2-bx = \square, \quad x^2-cx = \square \text{ .... } (1, 2, 3).$$

Assume $x^2-ax = (x-r)^2$; then $x = \dfrac{r^2}{2r-a}$. Substituting this in (2) and (3), and omitting the square factors, we shall have to find

$$r^2-2br+ab = \square, \quad r^2-2cr+ac = \square \text{ .... } (4, 5).$$

Assume $r^2-2br+ab = (r-s)^2$, then $r = \dfrac{s^2-ab}{2(s-b)}$. Substituting this result in (5), and omitting the square factors, it becomes

$$s^4-4cs^3+(4ac+4bc-4ab)\,s^2-4abcs+a^2b^2 = \square \text{ ....} (6).$$

Assume $(6) = (s^2+2cs-ab)^2$; then we obtain $s = \frac{1}{2}(a+b-c)$, from which we readily deduce $\quad x = \dfrac{(2ab+2ac+2bc-a^2-b^2-c^2)^2}{8\,(a+b-c)\,(a+c-b)\,(b+c-a)}$.

From this general expression any number of numerical answers may be found. In order that x may be positive, as is required, p and q must be taken so that the sum of any two of the numbers a, b, c must be greater than the third. If we take $p = 4$, $q = 3$, we shall find $a = \frac{1}{17}$, $b = \frac{1}{25}$, $c = \frac{1}{31}$, which values will give a positive, but very large, value for x. A smaller numerical answer may possibly be obtained by some different method of solution.

[If x be *negative*, the solution would give three square numbers in arithmetical progression, such that, if each be *increased* by its root, the results would all be rational squares.]

II. *Solution by* Asher B. Evans, M.A.

Let $\quad a^2x^2 = (2rs-r^2+s^2)^2x^2, \quad b^2x^2 = (r^2+s^2)^2x^2, \quad c^2x^2 = (2rs+r^2-s^2)^2x^2$
represent the three numbers. Then must

$$a^2x^2-ax = \square, \quad b^2x^2-bx = \square, \quad c^2x^2-cx = \square \text{ .... } (1, 2, 3).$$

Assume $(a-m)x$ for the root of (1), then $x = \dfrac{a}{2am-m^2}$. By substituting this value of x in (2) and (3), and multiplying the first result by $c^2\,(2am-m^2)^2$, and the second by $b^2\,(2am-m^2)^2$, we obtain

$$a^2b^2c^2-abc^2(2am-m^2) = \square = y^2, \quad a^2b^2c^2-ab^2c\,(2am-m^2) = \square = z^2 \text{ ...... } (4, 5).$$

(4) − (5) gives $\quad bc\,(2a-m)\,(ab-ac)\,m = (y+z)\,(y-z)$ .... (6).

Assume $\quad bc\,(2a-m) = y+z$, then $(ab-ac)\,m = y-z$;

and $\quad y = abc-\frac{1}{2}(ab+bc-ac)\,m, \quad z = abc-\frac{1}{2}(ab+bc-ac)\,m.$

By substituting this value of y in (4), or of z in (5), we find

$$m = \frac{4abc\,(ab-bc+ac)}{4ab^2c-(ab+bc-ac)^2} \text{ .... } (7).$$

In order to render ax, bx, cx positive, we must have a, c, x positive, since $b = r^2+s^2$ is necessarily positive. The condition that will render a and c positive, is that r be greater than s, and less than $(\sqrt{2}+1)\,s$. The con-

dition that will render x positive, is $m < 2a$, or $\dfrac{2bc\,(ab-bc+ac)}{4ab^2c-(ab+bc-ac)^2} < 1$.

For a numerical example let $s=3$ and $r=4$; then $a=17$, $b=25$, $c=31$,

$$m = \left(\frac{9327900}{864571}\right),\quad a^2x^2 = \left(\frac{12707211238697}{11011044931800}\right)^2,$$

$$b^2x^2 = \left(\frac{18687075351025}{11011044931800}\right)^2,\quad c^2x^2 = \left(\frac{23171973435271}{11011044931800}\right)^2.$$

[If the conditions $r > s$ and $< (\sqrt{2}+1)\,s$, and $\dfrac{2bc\,(ab-bc+ac)}{4ab^2c-(ab+bc-ac)^2} > 1$ be satisfied, a, b, c will be positive, and x negative; and in that case we shall have a solution to the problem referred to in the Editorial note to the foregoing solution.]

3150. (Proposed by the Rev. E. Hill, M.A.)—Across a valley is thrown a dam, converting it into a reservoir, whose surface varies as the square of the depth. Down the valley flows a stream of given velocity and section. If in the bottom of the dam be left a tunnel of small given section, find how long the dam will take to fill to a certain depth.

Solution by James Dale.

Let v and k be the constant velocity and section of the stream, k' the section of the tunnel, v' the velocity of the efflux at the time t and depth x, and A the area of surface of dam when the depth is x.

The water in the dam is increased in the time dt by $(kv-k'v')\,dt$;

therefore $$(kv-k'v')\,dt = \mathrm{A}dx = mx^2dx;$$

therefore $$dt = m\frac{x^2dx}{kv-k'v'} = m\frac{x^2dx}{kv-k'(2gx)^{\frac{1}{2}}},$$

and $$t = \frac{m}{k'(2g)^{\frac{1}{2}}}\int\frac{x^2dx}{\dfrac{kv}{k'(2g)^{\frac{1}{2}}}-x^{\frac{1}{2}}} = \frac{m}{k'(2g)^{\frac{1}{2}}}\int\frac{x^2dx}{a^{\frac{1}{2}}-x^{\frac{1}{2}}},\quad \left(\frac{kv}{k'(2g)^{\frac{1}{2}}} = a^{\frac{1}{2}}\right),$$

an integral which can be readily evaluated, and which is to be taken between the limits 0 and h of x.

3053. (Proposed by G. M. Minchin, B.A.)—Prove that

If $\left(\dfrac{4q^{\frac{1}{2}}}{kk'}\right)^{\frac{1}{6}}\left(\dfrac{\pi}{2\mathrm{K}}\right)^{\frac{1}{2}}.\,\Theta(q,x) = (1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)\ldots,$

then $\sin am\left\{\dfrac{2(1+k)\mathrm{K}x}{\pi}\quad\dfrac{2k^{\frac{1}{2}}}{1+k}\right\} = \left(\dfrac{(1+k)\pi}{2\mathrm{K}}\right)^{\frac{1}{2}}.\,\dfrac{\Theta^2(q,x)}{\Theta(q^{\frac{1}{2}},x)}.\sin am\dfrac{2\mathrm{K}x}{\pi}.$

Solution by the Proposer.

$$\mathrm{Sin}\,am\frac{2\mathrm{K}x}{\pi} = \frac{2q^{\frac{1}{4}}}{k^{\frac{1}{2}}}.\sin x\,\frac{(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)\ldots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)\ldots}.$$

Change q to $q^{\frac{1}{2}}$, and therefore k to $\dfrac{2k^{\frac{1}{2}}}{1+k}$, k' to $\dfrac{1-k}{1+k}$, K to $(1+k)$ K.

Then the numerator of the above fraction becomes

$$\left(\frac{4q^{\frac{1}{2}}}{kk'}\right)^{\frac{1}{6}}\left(\frac{\pi}{2\mathrm{K}}\right)^{\frac{1}{2}}.\ \Theta(q,\,x)\ .\ \sin x\,(1-2q^2\cos 2x+q^4)\,(1-2q^4\cos 2x+q^8)\ldots$$

$$=\left(\frac{4q^{\frac{1}{2}}}{kk'}\right)^{\frac{1}{6}}\cdot\frac{\pi}{2\mathrm{K}}\cdot\frac{k^{\frac{1}{2}}}{2q^{\frac{1}{4}}}\cdot\Theta^2(q,\,x)\ .\ \sin am\ .\ \frac{2\mathrm{K}x}{\pi},$$

while the denominator becomes

$$(1-2q^{\frac{1}{2}}\cos 2x+q)\,(1-2q^{\frac{3}{2}}\cos 2x+q^3)\ldots,$$

which we can see (by changing q, k, k', K, as above) to be

$$=\left\{\frac{2q^{\frac{1}{4}}(1+k)^2}{k^{\frac{1}{2}}(1-k)}\right\}^{\frac{1}{6}}\cdot\left(\frac{\pi}{2(1+k)\mathrm{K}}\right)^{\frac{1}{2}}\cdot\Theta(q^{\frac{1}{2}},\,x);$$

therefore $\qquad\qquad \sin am\left\{\dfrac{2(1+k)\mathrm{K}x}{\pi},\ \dfrac{2k^{\frac{1}{2}}}{1+k}\right\}$

$$=\frac{k^{\frac{1}{2}}(1+k)^{\frac{1}{2}}}{2^{\frac{1}{2}}q^{\frac{1}{4}}}\cdot\left(\frac{4q^{\frac{1}{2}}}{kk'}\right)^{\frac{1}{6}}\cdot\left(\frac{k^{\frac{1}{2}}(1-k)}{2q^{\frac{1}{4}}(1+k)^2}\right)^{\frac{1}{6}}\cdot\left(\frac{(1+k)\pi}{2\mathrm{K}}\right)^{\frac{1}{2}}\cdot\frac{\Theta^2(q,\,x)}{\Theta(q^{\frac{1}{2}},\,x)}\,.\ \sin\ am\ \frac{2\mathrm{K}x}{\pi}$$

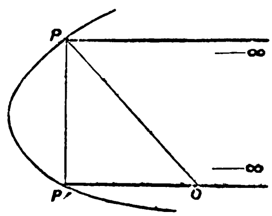
which reduces to the simple form

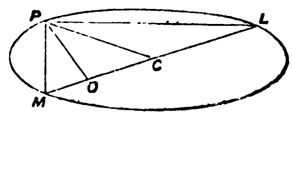
$$\left(\frac{(1+k)\,\pi}{2\mathrm{K}}\right)\cdot\frac{\Theta^2(q,\,x)}{\Theta(q^{\frac{1}{2}},\,x)}\ .\ \sin am\ \frac{2\mathrm{K}x}{\pi}.$$

It is easy to verify this equation when $x=\frac{1}{2}\pi$ or when $x=\frac{1}{4}\pi$. In the former case each side reduces to 1, and in the latter to $\left\{\frac{1}{2}(1+k)\right\}^{\frac{1}{2}}$, which may be seen at once by Lagrange's transformation.

2944. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—If O be that point in the normal to a parabola at P through which, if any chord pass, it will subtend a right angle at P, PO will be bisected by the axis.

Solution by the PROPOSER; R. W. GENESE; *and others.*



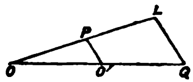


If P be the point (Fig. 1), and P∞, PP′ chords of the parabola, parallel and perpendicular to the axis, P′∞ will pass through the point O in question, whence PO is obviously bisected by the axis. So in any conic (Fig. 2), centre C, if PL, PM be chords parallel to the axes, LM will pass through O, whence CP, CO are equally inclined to the axes.

[Other Solutions have been given in the *Reprint*, Vol. XIII., p. 89.]

3098. (Proposed by the Editor.)—Given the inscribed and circumscribed circles of a triangle, show that the locus of the ortho-centre is a circle.

Solution by the Rev. J. Wolstenholme, M.A.



If O, O′ be the centres of the circumscribed and inscribed circles, P that of the nine-point circle, L the centre of perpendiculars, then P bisects OL; but because the nine-point circle touches the inscribed circle, $PO' = \frac{1}{2}R - r$, R, r being the given radii; hence, if OO′ be produced to Q, so that $O'Q = OO'$, $QL = 2O'P = R - 2r$, and the locus of L is a circle, centre Q.

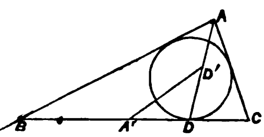
We can show immediately that the locus of the centres of the three escribed circles is a fixed circle; for if ABC be any one of the triangles, A′, B′, C′ the centres of the escribed circles, then O′ is the centre of perpendiculars and O the centre of the nine-point circle of the triangle A′B′C′. Hence, if O′O be produced to S, so that $O'O = OS$, S will be the centre of the circumscribed circle of A′B′C′, and is a fixed point; also the radius of the circumscribed circle is 2R, so that the circle is fixed.

[This theorem, due to Mr. Burnside, is proved by *Analysis*, for the general case of a conic, in Salmon's *Conics*, Art. 375, Ex. 6.]

3157. (Proposed by R. W. Genese.)—The circle inscribed in ABC touches BC in D: prove that the trilinear equation to the straight line joining the middle point of AD to middle point of BC is

$$(b-c)\,\alpha = b\beta - c\gamma.$$

I. *Solution by* W. H. Laverty, B.A.



If A′D′ be the line, the perpendiculars on it from B and C are evidently equal but of opposite signs; and p, q, r, being the three perpendiculars respectively from A, B, C, we have

$$\frac{p}{r} = \frac{\text{perpendicular from D}}{r} = \frac{A'D}{A'C} = \frac{\frac{1}{2}(c-b)}{\frac{1}{2}a}.$$

Now the equation to any straight line may be put into the form

$$pa\alpha + qb\beta + rc\gamma = 0;$$

therefore the equation to A′D′ is

$$(c-b)\,\alpha + b\beta - c\gamma = 0; \quad \text{or} \quad b\,(\alpha-\beta) = c\,(\alpha-\gamma).$$

II. *Solution by* S. Watson; J. Dale; *and others.*

Instead of the inscribed circle, take the conic $(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0$, and denote the middle points of AD, BC by $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$. Then if the triangle $ABC = \Delta$, obviously $a\alpha_1 = \Delta$; and β_1, γ_1 are half the values of β, γ derived from the equations $m\beta - n\gamma = 0$, and $b\beta + c\gamma = 2\Delta$,

hence $$\alpha_1 = \frac{\Delta}{a}, \quad \beta_1 = \frac{n\Delta}{bn+cm}, \quad \gamma_1 = \frac{m\Delta}{bn+cm} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

also $$\alpha_2 = 0, \quad \beta_2 = \frac{\Delta}{b}, \quad \gamma_2 = \frac{\Delta}{c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and the equation of a line through (1) and (2) is

$$a(cm-bn)\,\alpha + (b\beta - c\gamma)(cm+bn) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

In the particular case in the question, we have

$$\frac{a(cm-bn)}{cm+bn} = \frac{a(c\cos^2\frac{1}{2}\mathrm{B} - b\cos^2\frac{1}{2}\mathrm{C})}{c\cos^2\frac{1}{2}\mathrm{B} + b\cos^2\frac{1}{2}\mathrm{C}} = \frac{as(s_2-s_3)}{s(s_2+s_3)} = -(b-c),$$

and (3) becomes $(b-c)\,\alpha = b\beta - c\gamma$; or $b(\alpha-\beta) = c(\alpha-\gamma)$.

[The line evidently passes through the centre of the inscribed circle, and is perpendicular to the polar of A′ with respect to this circle.]

2865. (Proposed by J. J. Walker, M.A.)—Determine at what points on an ellipsoid the distance of the normal from the centre is a maximum.

Solution by the Proposer.

The equation to the ellipsoid being $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} = 1$, the square of the perpendicular from centre on the normal at the point (x, y, z) is equal to

$$x^2+y^2+z^2-u^{-1}, \text{ where } u = \frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4};$$

hence, for the points at which this is a maximum or minimum,

$$x\,dx + y\,dy + z\,dz + \left(\frac{x\,dx}{a^4}+\frac{y\,dy}{b^4}+\frac{z\,dz}{c^4}\right)u^{-2} = 0,$$

or $$\left(u^2+\frac{1}{a^4}\right)x\,dx + \left(u^2+\frac{1}{b^4}\right)x\,dy + \left(u^2+\frac{1}{c^4}\right)z\,dz = 0;$$

also $$\frac{x\,dx}{a^2}+\frac{y\,dy}{b^2}+\frac{z\,dz}{c^2} = 0;$$

$$\therefore \left(u^2+\frac{1}{a^4}+\frac{\lambda}{a^2}\right)x = 0, \quad \left(u^2+\frac{1}{b^4}+\frac{\lambda}{b^2}\right)y = 0, \quad \left(u^2+\frac{1}{c^4}+\frac{\lambda}{c^2}\right)z = 0.$$

Multiplying these equations by x, y, z respectively, and adding, we have

$$-\lambda = r^2u^2+u, \text{ where } r^2 = x^2+y^2+z^2;$$

and substituting this value for λ, they become

$$\left\{(a^2-r^2)u^2-u+\frac{1}{a^2}\right\}x = 0, \quad \left\{(b^2-r^2)u^2-u+\frac{1}{b^2}\right\}y = 0,$$

$$\left\{(c^2-r^2)u^2-u+\frac{1}{c^2}\right\}z = 0.$$

These equations are satisfied by $x=a$, $y=0$, $z=0$, or $x=0$, $y=b$, $z=0$, or $x=0$, $y=0$, $z=c$, which give the normals of minimum distance from the centre.

Eliminating u from each pair of the above equations, and reducing, there results the system

$$\left\{r^2-(b^2+c^2)+bc\right\} y^2z^2 = 0, \quad \left\{r^2-(a^2+c^2)+ac\right\} x^2z^2 = 0,$$
$$\left\{r^2-(a^2+b^2)+ab\right\} x^2y^2 = 0,$$

which are satisfied by the values

$z^2=0$, $r^2 = a^2+b^2-ab$, or $y^2=0$, $r^2=a^2+c^2-ac$, or $x^2=0$, $r^2=b^2+c^2-bc$;

otherwise written $z^2=0$, $x^2 = \dfrac{a^3}{a+b}$, $y^2 = \dfrac{b^3}{a+b}$, &c. &c.

At these points the normal is at the distances $a-b$, $a-c$, $b-c$, respectively, from the centre.

3077. (Proposed by R. W. GENESE.)—A′, B′, C′ are the middle points of the sides of the triangle ABC. Prove (1) that the line joining A′ to the pole of B′C′ with respect to the inscribed circle passes through the centre of the circle escribed in the angle A. Also (2) verify Faure's theorem that the triangle formed by the polars of A′, B′, C′ is equal in area to ABC.

Solution by JAMES DALE.

1. Let D, E, F be the points of contact of the inscribed circle; A″B″C″ the triangle formed by the polars of A′, B′, C′; and O the inscribed centre. Then DO, being perpendicular to B′C′, passes through its pole A″; and if DO cut B′C′ in P,

$$\text{DP} = \frac{\Delta}{a} \quad\text{and}\quad \text{DO} = \frac{\Delta}{s}, \quad\text{therefore}\quad \text{OP} = \frac{r\,(s-a)}{a};$$

$$\text{OA}'' = \frac{r^2}{\text{OP}} = \frac{ra}{s-a}, \quad\text{therefore}\quad \text{DA}'' = \frac{rs}{s-a} = r_1;$$

and if O′, D′ be the centre and point of contact of the escribed circle, it is evident, from the equality of the triangles A′DA″, A′D′O′, that A′A″ passes through O′.

2. Again, $\quad \text{OA}'' = \dfrac{ra}{s-a} = \dfrac{\Delta a}{s\,(s-a)} = a \tan \tfrac{1}{2}\text{A};$

similarly, $\quad \text{OB}'' = b \tan \tfrac{1}{2}\text{B}, \quad \text{OC}'' = c \tan \tfrac{1}{2}\text{C};$

also $\quad \text{B}''\text{OC}'' = 180^\circ - \text{A}, \quad \text{C}''\text{OA}'' = 180^\circ - \text{B}, \quad \text{A}''\text{OB}'' = 180^\circ - \text{C};$

$$\begin{aligned}\therefore\ \Delta\text{A}''\text{B}''\text{C}'' &= \tfrac{1}{2}(\text{OB}''.\text{OC}''\sin\text{A} + \text{OC}''.\text{OA}''\sin\text{B} + \text{OA}''.\text{OB}''\sin\text{C})\\ &= \tfrac{1}{2}(bc\sin\text{A}\tan\tfrac{1}{2}\text{B}\tan\tfrac{1}{2}\text{C} + \&\text{c.})\\ &= \Delta\text{ABC}(\tan\tfrac{1}{2}\text{B}\tan\tfrac{1}{2}\text{C} + \tan\tfrac{1}{2}\text{C}\tan\tfrac{1}{2}\text{A} + \tan\tfrac{1}{2}\text{A}\tan\tfrac{1}{2}\text{B})\\ &= \Delta\text{ABC}.\end{aligned}$$

3177. (Proposed by Rev. A. F. TORRY, M.A.)—An ellipse and an hyperbola are confocal, and at the common points the tangents to the ellipse are parallel to the asymptotes of the hyperbola; show that the axes of the ellipse are in the duplicate ratio of those of the hyperbola.

Solution by R. TUCKER, M.A.

Since the ellipse $\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2}=1$ and hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ are confocal, we have $$a'^2-b'^2=a^2+b^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

If now (h, k) be one of the common points, we have also $\dfrac{b'^2}{a'^2}\dfrac{h}{k}=\dfrac{b}{a}\ldots(2)$; and from the two curves we have

$$\left(\frac{b^2}{a^2}+\frac{b'^2}{a'^2}\right)h^2=b^2+b'^2,\quad \left(\frac{a'^2}{b'^2}+\frac{a^2}{b^2}\right)k^2=a'^2-a^2\;\ldots\ldots\ldots\;(3),$$

or $$\frac{h^2}{k^2}=\frac{b^2a'^4}{a^2b'^4}\text{ by (2)},\quad\text{and}\quad \frac{h^2}{k^2}=\frac{a^2a'^2}{b^2b'^2}\text{ by (3) and (1)};$$

that is, $\dfrac{a'}{b'}=\dfrac{a^2}{b^2}$, which proves the property stated.

3141. (Proposed by S. WATSON.)—Through the focus of an ellipse two chords are drawn at right angles to each other. Find (1) the average, (2) the maximum, and (3) the minimum area of the quadrilateral formed by joining their extremities.

I. *Solution by* ASHER B. EVANS, M.A.

1. Let AB, CD be two focal chords at right angles to each other; a, b the semi-axes; $c^2=a^2-b^2$; e the eccentricity; θ the inclination of AB to the major axis; H the area of the quadrilateral ADBC; A_1 the maximum, A_2 the mean or average, and A_3 the minimum area. Then we have

$$AB=\frac{2b^2}{a}\left(\frac{1}{1-e^2\cos^2\theta}\right),\quad CD=\frac{2b^2}{a}\left(\frac{1}{1-e^2\sin^2\theta}\right);$$

$$H=\tfrac{1}{2}AB.CD=\frac{2b^4}{a^2}\left(\frac{1}{1-e^2+\frac{1}{4}e^4\sin^2 2\theta}\right)=\frac{8a^2b^4}{4a^2b^2+c^4\sin^2 2\theta}\;\ldots\ldots\;(1);$$

$$A_2=4a^2b^4\int_0^{\frac{1}{2}\pi}\frac{d(2\theta)}{4a^2b^2+c^4\sin^2 2\theta}\div\int_0^{\frac{1}{2}\pi}d\theta=\frac{4ab^3}{a^2+b^2}.$$

2. The quadrilateral ADBC will evidently be a maximum when $4a^2b^2+c^4\sin^2 2\theta$ is a minimum, that is, when $\theta=0$ or $\frac{1}{2}\pi$; and the maximum area is $A_1=2b^2$.

3. The quadrilateral is a minimum when $\theta=\frac{1}{4}\pi$, and its area is then

$$A_3=\frac{8a^2b^4}{(a^2+b^2)^2}.$$

II. *Solution by* JAMES DALE.

Let l_1, l_2 be two focal chords making angles θ, and $\frac{1}{2}\pi+\theta$ with the axis-major, then $2p$ being the latus rectum, we have

$$l_1 = \frac{2p}{1-e^2\cos^2\theta}, \quad l_2 = \frac{2p}{1-e^2\sin^2\theta},$$

therefore
$$\mathrm{H} = \tfrac{1}{2}l_1 l_2 = \frac{2p^2}{1-e^2+\frac{1}{4}e^4\sin^2 2\theta}.$$

Hence, the area is a maximum when $\theta = 0$ or $\frac{1}{2}\pi$, that is, when one chord is the major axis, and the other the latus rectum. The area is a minimum when $\theta = \frac{1}{4}\pi$.

$$\text{The average area } (\mathrm{A}_2) = \frac{1}{\frac{1}{4}\pi}\int_0^{\frac{1}{4}\pi} \frac{2p^2\,d\theta}{(1-e^2)+\frac{1}{4}e^4(\sin 2\theta)^2}$$

$$= \frac{4p^2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{d(\tan 2\theta)}{1-e^2+(1-\frac{1}{2}e^2)^2(\tan 2\theta)^2} = \frac{2p^2}{(1-\frac{1}{2}e^2)(1-e^2)^{\frac{1}{2}}}.$$

$$\text{maximum area} = \mathrm{A}_1 = 2b^2;$$

$$\text{average} \quad ,, \quad = \mathrm{A}_2 = 2b^2 . \frac{2ab}{a^2+b^2};$$

$$\text{minimum} \quad ,, \quad = \mathrm{A}_3 = 2b^2 . \frac{4a^2b^2}{(a^2+b^2)^2}.$$

[It will be observed that the average area (A_2) is a mean proportional between the greatest and least areas (A_1, A_3).

2986. (Proposed by C. W. MERRIFIELD, F.R.S.)—Two diameters of an oval bisect one another at right angles. Supposing the diameters given, required the form of the oval for which the ratio of the area to the perimeter is the greatest possible.

Solution by R. W. GENESE.

With the usual notation, viz., θ = inclination of p (perpendicular on tangent) to initial line, and $p'' = \frac{d^2p}{d\theta^2}$, we have

$$\mathrm{P} = \text{perimeter} = \int (p+p'')\,d\theta, \quad \text{and} \quad \mathrm{A} = \text{area} = \int p\,ds = \int (p+p'')\,p\,d\theta.$$

Now $\frac{\mathrm{A}}{\mathrm{P}}$ is to be a maximum, therefore $\mathrm{P}\delta\mathrm{A} = \mathrm{A}\delta\mathrm{P}$, or, by the method of the Calculus of Variations,

$$\mathrm{P}\int \left\{(2p+p'')+p''\right\} d\theta\,\delta p = \mathrm{A}\int d\theta\,\delta p;$$

therefore
$$p+p'' = \frac{\mathrm{A}}{2\mathrm{P}} \text{ always.}$$

But $p+p''$ = radius of curvature, therefore the oval is formed by circular arcs.

NOTE ON QUESTION 2986. BY C. W. MERRIFIELD, F.R.S.

With regard to Mr. Genese's Solution of this question, I think the form of my question is a little at fault. I ought to have said "two diameters of a *convex* oval." The necessity for the limitation did not occur to me until I saw the solution.

Mr. Genese's Solution is not quite complete as it stands. There is, in reality, no finite maximum, if the oval be not limited by the condition of convexity; if it be so limited, the solution is not a circular arc, or any combination of such arcs. To show this, let us take the simplest case in which the given diameters are equal. The circle described upon them and the circumscribed square will both give the same value for the ratio of area to perimeter. The form for which this ratio is a maximum or a minimum will therefore lie between the two, and it is easily shown arithmetically that it is the maximum form which is thus intermediate.

I am not prepared myself with a solution of the problem restricted as above stated, but I am not sufficiently well versed in the calculus of variations to assume that what is difficult to me is so to others.

The problem is of interest as giving the most economical form of midship section for a ship under certain circumstances.

Mr. Genese's Solution is correct under the conditions of a *general* maximum obtained by the calculus of variations; but the absolute maximum appears to be *infinity* under his conditions, and it does not appear to be circles, as restricted according to my intention.

3007. (Proposed by R. TUCKER, M.A.)—Ellipses are described having a common transverse axis, and the osculating circles at the extremities of the latera recta are drawn; show that the centres of these circles lie on a septic passing through the centre and extremity of the axis.

Solution by the PROPOSER.

The general coordinates of the centre of curvature are

$$\alpha = ae^2 \cos^3 \phi, \quad \beta = -\frac{a^2e^2}{b} \sin^3 \phi;$$

but, in this case, $\cos \phi = e, \quad b = a \sin \phi,$

whence the locus required is found by eliminating e between

$$\alpha = ae^5, \quad \beta = -ae^3(1-e^2),$$

that is, $e^5 = \frac{\alpha}{a} = \lambda, \quad e^4 - e^2 = \frac{\beta}{a} = \mu.$

Dividing, $\frac{e^3}{e^2-1} = \frac{\lambda}{\mu} = \nu.$ Our equations now are

$$e^4 - e^2 - \mu = 0, \quad e^3 - \nu e^2 + \nu = 0, \quad e^3 + \lambda e - \nu = 0,$$

which readily reduce to

$$(m^2-1)e^2 - \mu e - \lambda - \mu^2 = 0, \quad \mu e^2 + \lambda e - \mu - \nu = 0.$$

The result is $\lambda^3 + \mu^2 + \mu^4\nu + 5\nu^2 - 5\mu^3\nu + 5\mu\nu - 6\mu^2\nu^2 - \mu^4 - \lambda\nu^2 = 0,$

or, in terms of α, β,

$$\beta^7 + a^3\alpha^2\beta^2 + a^2\alpha^5 + 5a\alpha^2\beta^4 - 5a^2\alpha^4\beta + 5a^2\alpha^2\beta^3 - 6a\alpha^4\beta^2 - a^3\alpha^4 - \alpha^2\beta^5 = 0,$$

the equation to the septic required.

2790. (Proposed by A. MARTIN.)—Find the mean distance of all the points in a right circular cylinder from one end of the axis.

Solution by the PROPOSER.

Let a be the length of the cylinder, r the radius of its base, (ρ, ϕ) polar coordinates of any point within the cylinder, and M the mean distance required. Then

$$M = \frac{2\pi\iint \rho^3 \sin\phi \, d\rho \, d\phi}{\pi a r^2} = \frac{2}{ar^2}\iint \rho^3 \sin\phi \, d\rho \, d\phi = \frac{1}{2ar^2}\int \rho^4 \sin\phi \, d\phi$$

$$= \frac{a^3}{2r^2}\int_0^{\cos^{-1}\frac{a}{(a^2+r^2)^{\frac{1}{2}}}} \frac{\sin\phi \, d\phi}{\cos^4\phi} + \frac{r^2}{2a}\int_{\frac{1}{2}\pi}^{\cot^{-1}\frac{a}{r}} \frac{d\phi}{\sin^3\phi}$$

$$= \frac{(2a^2+5r^2)(a^2+r^2)^{\frac{1}{2}}}{12r^2} - \frac{a^3}{6r^2} + \frac{r^2}{4a}\log\left(\frac{a+(a^2+r^2)^{\frac{1}{2}}}{r}\right).$$

[Another Solution is given on p. 52 of Vol. XII. of the *Reprint.*]

3074. (Proposed by the Rev. J. BLISSARD.)—To prove that

$$\Gamma(n+1) = n + a_1 n(n-1) + a_2 n(n-1)(n-2) + \ldots + a_r n(n-1)\ldots(n-r) + \ldots,$$

where $$a_r = 1 - 1 + \frac{1}{1.2} - \frac{1}{1.2.3} + \ldots \pm \frac{1}{1.2\ldots r},$$

that is, = the sum of the first $r+1$ terms of the expansion of ϵ^{-1}.

3103. (Proposed by the Rev. J. BLISSARD.)—To prove that

$$\frac{1}{\Gamma(n+1)} = 1 - \frac{1}{2}\,\frac{n(n-1)}{1.2} + \frac{2}{3}\,\frac{n(n-1)(n-2)}{1.2.3} - \ldots\ldots$$
$$\ldots\ldots + (-1)^r a_r \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r},$$

where $$a_r = 1 - \frac{r}{1^2} + \frac{r(r-1)}{(1.2)^2} - \frac{r(r-1)(r-2)}{(1.2.3)^2} + \ldots\ldots$$

Solution by the PROPOSER.

The above formulæ are particular cases of a general theorem, readily obtained by use of "Representative" Notation, as follows:—

Let $$fn = u^n = u^m . u^{n-m} = u^m\{1+(u-1)\}^{n-m}$$

$$= u^m\left\{1 + \frac{n-m}{1}(u-1) + \frac{(n-m)(n-m-1)}{1.2}(u-1)^2 + \ldots\ldots\right\}$$

$$= u^m + \frac{n-m}{1}(u^{m+1} - u^m) + \frac{(n-m)(n-m-1)}{1.2}(u^{m+2} - 2u^{m+1} + u^m) + \ldots\ldots,$$

that is, $$fn = fm + \frac{n-m}{1}\{f(m+1) - fm\}$$

$$+ \frac{(n-m)(n-m-1)}{1.2}\{f(m+2) - 2f(m+1) + fm\} + \&c.\ldots\ldots\ldots(B).$$

Ex. 1: Let $fn = \Gamma n$, and let $m=1$; then, from (B),

$$\Gamma n = \Gamma 1 + \frac{n-1}{1}(\Gamma 2 - \Gamma 1) + \frac{(n-1)(n-2)}{1.2}(\Gamma 3 - 2\Gamma 2 + \Gamma 1) + \dots$$

$$\dots + \frac{(n-1)(n-2)\dots(n-r)}{1.2\dots r}\left(\Gamma(r+1) - r.\Gamma r + \frac{r(r-1)}{1.2}\Gamma(r-1) - \dots\right) + \dots;$$

$$\therefore\ \Gamma(n+1)\,(=n\Gamma n) = n + n(n-1)(1-1) + n(n-1)(n-2)\left(1-1+\frac{1}{1.2}\right) + \dots$$

$$\dots + n(n-1)\dots(n-r)\left(1-1+\frac{1}{1.2}-\frac{1}{1.2.3}+\dots\pm\frac{1}{1.2\dots r}\right)+\dots,$$

which is the solution of Question 3074.

Ex. 2: Let $fn = \dfrac{1}{\Gamma n+1}$, and let $m=0$; then, from (B),

$$\frac{1}{\Gamma(n+1)} = \frac{1}{\Gamma 1} + n\left(\frac{1}{\Gamma 2} - \frac{1}{\Gamma 1}\right) + \frac{n(n-1)}{1.2}\left(\frac{1}{\Gamma 3} - \frac{2}{\Gamma 2} + \frac{1}{\Gamma 1}\right) + \dots$$

$$= 1 + \frac{n(n-1)}{1.2}\left(1-2+\frac{1}{1.2}\right) - \frac{n(n-1)(n-2)}{1.2.3}\left(1-3+\frac{3}{1.2}-\frac{1}{1.2.3}\right)+\dots,$$

where the general term is

$$(-1)^r\,\frac{n(n-1)\dots(n-r+1)}{1.2\dots r}\left\{1-\frac{r}{1^2}+\frac{r(r-1)}{(1.2)^2}-\frac{r(r-1)(r-2)}{(1.2.3)^2}+\dots\right\},$$

which is the solution of Question 3103.

NOTE.—The theorem (B) is evidently an identity when n and m are positive integers. The advantage of it is, that it appears to hold good generally.

3030. (Proposed by R. TUCKER, M.A.)—Solve the equation

$$x^3 + px^2 + qx + r = 0,$$

when (1) ...... $q^3 - 4pqr + 8r^2 = 0$, *Ex.*: $x^3 + \frac{253}{80}x^2 + 5x + 4 = 0$;
and (2) .... $p^3 - 4pq + 8r = 0$, *Ex.*: $x^3 + 3x^2 + 2x - \frac{3}{8} = 0$.

Solution by A. A. BOURNE; J. J. WALKER, M.A.; R. W. GENESE; *the* PROPOSER; *and many others.*

Taking first the solution when the coefficients are connected by the identity (2), and eliminating r, there results

$$(2x)^3 + 2p(2x)^2 + 4q(2x) - p^3 + 4pq = 0,$$

or
$$(2x)^2(2x+p) + p\left\{(2x)^2 - p^2\right\} + 4q(2x+p) = 0,$$

or
$$\left\{(2x)^2 + p(2x-p) + 4q\right\}(2x+p) = 0,$$

whence it appears that one root is $x = -\frac{1}{2}p$, and the other two are given by the quadratic $\quad 4x^2 + 2px - p^2 + 4q = 0.$

In the numerical example, the roots are $-\frac{3}{2}$ and $\frac{1}{4}(-3 \pm \sqrt{13})$.

Case (1) is that in which the coefficients of the equation whose roots are

the reciprocals of those of $x^3+px^2+qx+r=0$ satisfy a similar relation to that given in (2); consequently $\dfrac{1}{x}=\dfrac{-q}{2r}$, or $x=-\dfrac{2r}{q}$.

In the numerical example, the roots are $-\frac{4}{5}$ and $\frac{1}{32}(-25\pm\sqrt{-1935})$.

3205. (Proposed by Sir JAMES COCKLE, F.R.S.)—Solve the linear partial differential equation of the second order

$$2(xr-t)+mp=0 \quad\text{.................................}(1)$$

for the cases in which $m=1$ and $m=3$.

Solution by the PROPOSER.

Monge's system for the solution of (1) is (Boole's *Diff. Eq.*, p. 369)

$$2x\,dy^2-2dx^2=0,\quad 2x\,dp\,dy-2dq\,dx+mp\,dx\,dy=0 \text{ ...... } (2), (3),$$

combined, if needful, with $\quad dz=pdx+qdy$ ....(4).

From (2) we deduce $\quad dy=\dfrac{dx}{\sqrt{x}},\quad y-2\sqrt{x}=a$ ....(5, 6),

the radical being susceptible of either sign (*i. e.* + or −). Eliminate dy from (3) by means of (5), and we have, after dividing by dx,

$$2\sqrt{x}\,dp-2dq+mp\frac{dx}{\sqrt{x}}=0 \quad\text{........................}(7).$$

First, let $m=1$. Then (7) becomes $\;2d(p\sqrt{x}-q)=0$ .... (8),

whence $\quad p\sqrt{x}-q=b$ ....(9),

where b, like a, is an arbitrary constant. Hence, combining (6) and (9),

$$p\sqrt{x}-q=f(y-2\sqrt{x}) \quad\text{..........................}(10),$$

where f is an arbitrary function, is a first integral of (1) when $m=1$. Indeed, (10) gives two first integrals, for we may affect the radical with either sign.

Secondly, let $m=3$. Then (7), multiplied into $\sqrt{x}$, gives

$$2(x\,dp+p\,dx)-2\sqrt{x}\,dq+p\,dx=0 \quad\text{........................}(11),$$

and (4), combined with (5), gives $\quad -dz+p\,dx+q\dfrac{dx}{\sqrt{x}}=0$ .... (12).

(11)−(12) gives $\;2(xdp+pdx)+dz-\left(2\sqrt{x}\,dq+q\dfrac{dx}{\sqrt{x}}\right)=0$ ....(13),

or $\quad 2d(xp)+dz-2d(q\sqrt{x})=0$ ....(14);

whence, by integration, $\quad 2px+z-2q\sqrt{x}=b$ ....(15).

Consequently, $\quad 2px+z-2q\sqrt{x}=f(y-2\sqrt{x})$ ....(16)

gives, by varying the radical sign, two first integrals.

Thirdly, these equations afford an illustration of the following rule, which may yield results where Monge's method fails:

Rule.—In Monge's second auxiliary equation, make either p or q a constant. Integrate the resulting system in the form $u=b$. Denote by

$dy = ndx$ that form of the first auxiliary which is employed in the integration, and by $dy = \mathrm{N}dx$ the other form. Let the integral of $dy = ndx$ be $v = a$. Then, if
$$\frac{du}{dq} = \mathrm{N}\frac{du}{dp} \quad \text{.................................(17)},$$
the equation $u = f(v)$, wherein f is arbitrary, will be a first integral of the given equation.

Scholium.—The advantage of the rule is this. Even when (17) is not satisfied, still, if u satisfies a relation of the form
$$\frac{du}{dq} - \mathrm{N}\frac{du}{dp} = \mathrm{U}_2\left(\frac{d\mathrm{U}}{dq} - \mathrm{N}\frac{d\mathrm{U}}{dp}\right) \quad \text{....................(18)},$$
then $\mathrm{U}_2 = 0$ is a first integral of the given equation.

Example.—The equation $r + 2zs + (z^2 - \mathrm{A}^2)\,t + 2q\,(p + qz) = 0$ ....(19) yields to the method of the Scholium, but does not yield to Monge's method.

Applying the rule to (3), let $m = 1$, and make p constant. Then (3), divided by dx, becomes $0 = -2dq + pdy = -2dq + p\dfrac{dx}{\sqrt{x}}$.

Hence, integrating as if p were constant,
$$u = -2q + 2p\sqrt{x} = b.$$

Now taking $n = \dfrac{1}{\sqrt{x}}$, and therefore $\mathrm{N} = -\dfrac{1}{\sqrt{x}}$,

we have
$$\frac{du}{dq} = -2 = -2\frac{\sqrt{x}}{\sqrt{x}} = \mathrm{N}\frac{du}{dp} \quad \text{...................... (20)}.$$
Consequently (17) is satisfied and (10) is the final solution.

Next, let $m = 3$, and make q constant. Then (3) becomes, on dividing by dy, $0 = 2\,(xdp + pdx) + pdx = 2d\,(xp) + dz - qdy = d\,(2xp + z) - q\dfrac{dx}{\sqrt{x}}$.

Hence, integrating as if q were constant,
$$u = 2xp + z - 2q\sqrt{x} = b,$$
and, as before, we have (17) satisfied, for
$$\frac{du}{dq} = -2\sqrt{x} = -2\frac{x}{\sqrt{x}} = \mathrm{N}\frac{du}{dp},$$
and consequently (16) is the solution.

The investigation upon which the Rule and Scholium are based will be found in section 3 of my second paper "On the Motion of Fluids," in the *Quarterly Journal* (see No. 40, 1870).

3120. (Proposed by Professor Cayley.)—To find the equation of the Jacobian of the quadric surfaces through the six points
$$(1, 0, 0, 0),\ (0, 1, 0, 0),\ (0, 0, 1, 0),\ (0, 0, 0, 1),\ (1, 1, 1, 1),\ (\alpha, \beta, \gamma, \delta).$$

Solution by the Proposer.

Writing for shortness
$$a = \beta - \gamma,\quad b = \gamma - \alpha,\quad c = \alpha - \beta,\quad f = \alpha - \delta,\quad g = \beta - \delta,\quad h = \gamma - \delta,$$

(so that $a+h-g=0$, &c., $a+b+c=0$, $af+bg+ch=0$,) the six points lie in each of the plane pairs

$$x\,(hy-gz+aw)=0,\quad y\,(-hx+fz+bw)=0,$$
$$z\,(gx-fy+cw)=0,\quad w\,(-ax-by-cz)=0.$$

We cannot take these as the four quadrics, on account of the identical equation $0=0$, which is obtained by adding the four equations; but we may take the first three of them for three of the quadrics, and for the fourth quadric the cone, vertex $(0,0,0,1)$, which passes through the other five points; viz., this is $a\alpha yz+b\beta zx+c\gamma xy=0$. We write therefore

$$\mathrm{P}=x\,(hy-gz+aw),\quad \mathrm{Q}=y\,(-hx+fz+bw),$$
$$\mathrm{R}=z\,(gx-fy+cw),\quad \mathrm{S}=a\alpha yz+b\beta zx+c\gamma xy;$$

and equate to zero the determinant formed with the derived functions of P, Q, R, S in regard to the coordinates (x, y, z, w) respectively. If, for a moment, we write A, B, C to denote $bg-ch$, $ch-af$, $af-bg$ respectively, it is easily found that the term containing $d\mathrm{S}$ is

$$(b\beta z+c\gamma y)\,x\left(-agh,\ bhf,\ cfg,\ abc,\ -af^2,\ -g\mathrm{B},\ h\mathrm{C},\ a\mathrm{A},\ b^2g,\ -c^2h\,\mathbin{\big\langle\!\big\rangle}\, x, y, z, w\right)^2,$$

the terms containing $d_y\mathrm{S}$ and $d_z\mathrm{S}$, are derived from this by a mere cyclical interchange of the letters (x, y, z), (A, B, C), (a, b, c), and (f, g, h); collecting and reducing, it is found that the whole equation divides by $2abc$; and that, omitting this factor, the result is

$$\left.\begin{array}{l} ayz\,(\alpha w^2-\delta x^2)+fxw\,(\beta z^2-\gamma y^2) \\ +\,bzx\,(\beta w^2-\delta y^2)+gyw\,(\gamma x^2-\alpha z^2) \\ +\,cxy\,(\gamma w^2-\delta z^2)+hzw\,(\alpha y^2-\beta x^2) \end{array}\right\}=0,$$

which, substituting for (a, b, c, f, g, h) their values, is the required form.

If, in the equation, we write for instance $x=0$, the equation becomes

$$ayzw\,(hy-gz+aw)=0;$$

or, the section by the plane is made up of four lines. Calling the given points 1, 2, 3, 4, 5, 6, it thus appears that the surface contains the fifteen lines 12, 13, ... 56, and also the ten lines 123·456, &c., in all twenty-five lines. Moreover, since the surface contains the lines 12, 13, 14, 15, 16, it is clear that the point 1 is a node (conical point) on the surface; and the like as to the points 2, 3, 4, 5, 6.

3207. (Proposed by Professor SYLVESTER.)—If $(a_1, a_2, \dots a_n)$, $(b_1, b_2, \dots b_n)$ be two sets of quantities all distinct from one another; show that in order to the co-existence of the $2n$ equations of the form

$$(a_i-b_1)(a_i-b_2)\,\dots\dots\,(a_i-b_n)=\mathrm{A}_i\,(a_i-a_1)(a_i-a_2)\dots\dots$$
$$(b_i-a_1)(b_i-a_2)\,\dots\dots\,(b_i-a_n)=\mathrm{B}_i\,(b_i-b_1)(b_i-b_2)\dots\dots\ .$$

we must have $\qquad \Sigma_n^1\mathrm{A}_i+\Sigma_n^1\mathrm{B}_i=0.$

Solution by C. W. MERRIFIELD, F.R.S.

1. $\Sigma\mathrm{A}$ is integral. For if we reduce all the fractions of which it con-

sists to a common denominator, which will contain $\frac{1}{2}n(n-1)$ factors, we may divide them into the following two classes:—

(*a*.) Those which contain any given factor such as $a_k - a_l$ explicitly.

(*b*.) Those containing only one of a_k and a_l; these occur in such a way as to cancel one another when we make $a_k = a_l$. Now this, although against hypothesis, is admissible for the mere purpose of showing that there must be a factor of the form $a_k - a_l$.

It follows that the sum of the numerators of the fractions, when reduced to a common denominator, contains $a_k - a_l$ as a factor, k and l being any integers from 1 to n, and as all the a's are different, it must contain the whole denominator as a factor.

[To make this clear, let $n = 3$; then $(a_1-a_2)(a_2-a_3)(a_3-a_1)\Sigma A =$

$$-(a_2-a_3)(a_1-b_1)(a_1-b_2)(a_1-b_3)$$
$$-(a_3-a_1)(a_2-b_1)(a_2-b_2)(a_2-b_3)$$
$$-(a_1-a_2)(a_3-b_1)(a_3-b_2)(a_3-b_3).$$

The first term on the right hand side contains $a_2 - a_3$ explicitly. The next two terms cancel one another if $a_2 = a_3$, for they then become identical except as to sign. Taken together, therefore, they contain the factor $(a_2 - a_3)$. Similarly it may be shown that the right hand side contains the factors $(a_3 - a_1)$ and $(a_1 - a_2)$. Hence ΣA is integral.]

2. It is evident that ΣA is of the first degree in terms of $a_1\, a_2$ &c., $b_1\, b_2$ &c.

3. ΣA will therefore take the form

$$\lambda_1 a_1 + \lambda_2 a_2 + \ldots\ldots - \mu_1 b_1 - \mu_2 b_2 - \ldots\ldots$$

It is clear from symmetry that all the λ's must be equal, and all the μ's equal.

4. By examining simple cases we find $\lambda = \mu$. Indeed, *except as to sign*, this is evident also from symmetry.

[This is easily verified in the case of $n = 1$, when we get

$$A = a_1 - b_1, \quad B = b_1 - a_1;$$

and in the case of $n = 2$, when we obtain

$$A_1 + A_2 = a_1 + a_2 - (b_1 + b_2), \quad B_1 + B_2 = b_1 + b_2 - (a_1 + a_2).]$$

It follows that $\Sigma A + \Sigma B = 0$, and also that $\Sigma A = \Sigma a - \Sigma b$.

[The Proposer remarks that this system of equations occurs in the Diophantine question connected with the theory of involutes to a circle.]

Note on Question 3232. By G. O. Hanlon.

This theorem leads to some consequences which may not be uninteresting. Let l be the major axis of the ellipse, $OA = r$ and $OB = r'$, then $O'A = l - r$ and $O'B = l - r'$. Let h and h' be the perpendiculars from O' on the tangents; the theorem then gives

$$\frac{h^2}{h'^2} = \frac{r'(l-r)}{r(l-r')} = \frac{l-r}{r} + \frac{l-r'}{r'}.$$

Let us suppose that the ellipse represents the orbit of a planet with the sun at O; then since the velocity of a planet varies inversely as the per-

pendicular from the sun on its direction, it varies directly as the perpendicular from the empty focus on its direction, therefore if v and v' represent the velocities at r and r' we have

$$\frac{h^2}{h'^2} = \frac{v^2}{v'^2} = \frac{l-r}{r} + \frac{l-r'}{r'}.$$

Now, since r' cannot enter into the expression for v^2, nor r into that for v'^2, we must have

$$v^2 = k\,.\frac{l-r}{r},$$

where k is some constant. Divide by l to make this expression homogeneous, and we get

$$v^2 = k'\left\{\frac{1}{r} - \frac{1}{l}\right\},$$

which is an interesting integration for the planets of the equation

$$v^2 = -2\int \mathrm{F}dr,$$

and which, written in the form

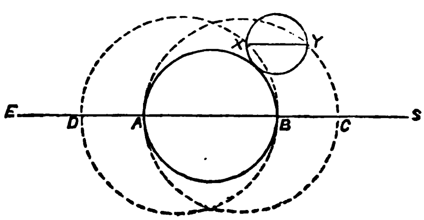
$$v^2 = k\frac{l-r}{r},$$

shows that *the square of the velocity of a planet varies as the ratio of its distances from the two foci.*

From the construction in the theorem, it is easy to see that the velocities of a planet *at equal distances from either axis* are inversely as the distances from the sun, of which general theorem the relation between the velocities at aphelion and perihelion is only a particular case. This becomes evident when we consider that the circular velocity of a planet at any point varies inversely as the focal distance, and that if two circles are described from the same focus of an ellipse cutting the curve in points at equal distances from either axis, they cut at the same angle.

3198. (Proposed by the Rev. F. D. Thomson, M.A.)—A polished wire of circular section bent into the form of a circle rolls rapidly on the outside of a fixed circle in the same horizontal plane which is exposed to the rays of the sun. Show that a spectator at a considerable distance on the side remote from the sun will see a bright curve consisting of two equal circles touching the fixed circle.

Solution by the Rev. R. Townsend, M.A., F.R.S.

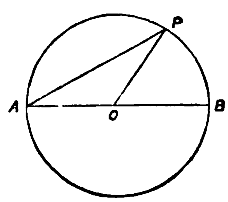


Let AB be the diameter of the fixed circle in the vertical plane passing through the positions E and S of the eye and sun, and XY the parallel diameter of the revolving circle in any position; then, X and Y being the only points of the latter circle for which a ray of light from S could

be reflected to E, the two circles (represented by dotted lines in the figure) on AC and BD as diameters, where AC = BD = AB+XY, which are evidently the loci of X and Y, will consequently, for a sufficiently rapid velocity of revolution, be the appearance presented at E.

3187. (Proposed by A. MARTIN.)—A chord is drawn at random across a given circle, and then two points are taken at random in the surface. Find the chance that both points are on the same side of the chord.

Solution by STEPHEN WATSON.

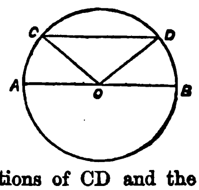


1. Let the chord AP be that formed by joining two points A and P taken at random in the circumference, one of which, as A, may be taken as fixed, and draw the diameter AB. Put AB = $2a$, $\angle$ PAB = ϕ, and denote the segments on each side of AP by u, u_1. Then

$$u = a^2\left(\tfrac{1}{2}\pi - \phi - \sin\phi\cos\phi\right),$$
$$u_1 = a^2\left(\tfrac{1}{2}\pi + \phi + \sin\phi\cos\phi\right).$$

The number of ways both points can lie on the same side of AP is $u^2+u_1^2$, an element at P is $ad(2\phi)$, the limits are ϕ from 0 to $\frac{1}{2}\pi$ and the result doubled for P below B, and the total number of positions of P and the two points is $2a\pi \times a^4\pi^2 = 2a^5\pi^3$; hence, the required chance in this case is

$$\frac{4a}{2a^5\pi^3}\int_0^{\frac{1}{2}\pi}(u^2+u_1^2)\,d\phi = \frac{2}{3}+\frac{5}{4\pi^2}.$$



2. Let now the chord CD be that formed by a line drawn in a random direction cutting the circle in CD. Draw the diameter AB parallel to CD, and join OC, OD, (O being the centre). Put OC = a, $\angle$OCD = ϕ; then u and u_1 are as before, the limits are ϕ from 0 to $\frac{1}{2}\pi$, and doubled; and since CD is as likely to be parallel to one diameter of the circle as another, we only need take into account the positions of CD parallel to AB; hence, the total positions of CD and the two points is $2a \times a^4\pi^2 = 2a^5\pi^2$, and the chance in this case is,

$$\frac{2}{2a^5\pi^2}\int_0^{\frac{1}{2}\pi}(u^2+u_1^2)\,d\,(a\sin\phi) = 1 - \frac{128}{45\pi^2}.$$

3161. (Proposed by J. B. SANDERS.)—A ladder rests against a vertical wall, to which it is inclined at an angle of 45°; the coefficients of friction of the wall and horizontal plane being respectively $\frac{1}{3}$ and $\frac{1}{2}$, and the centre of gravity of the ladder being at its middle round. A man whose weight is half the weight of the ladder begins to ascend it: find to what height he will go before the ladder begins to slide.

Solution by James Dale.

Let W = weight of ladder, $\frac{W}{n}$ = weight of man, l = length of ladder, x = distance of man from lower end when ladder slips, R_1 = reaction of wall, R_2 = reaction of plane.

Resolving horizontally, $\qquad R_1 = \frac{1}{2} R_2.$

Resolving vertically, $\quad W + \frac{W}{n} = R_2 + \frac{1}{3} R_1 = \frac{7}{3} R_1.$

Taking moments round the lowest point,

$$W\left(\frac{l}{2} + \frac{x}{n}\right) = l(R_1 + \frac{1}{3}R_1) = \frac{4}{3} l R_1 = \frac{4}{7} \cdot \frac{n+1}{n} l W;$$

therefore $$x = \frac{n+8}{14} l.$$

When, as in the given case, $n = 2$, we have $x = \frac{5}{7} l$.

When $n = 6$, we have $\qquad x = l$,

that is, a man whose weight is one-sixth the weight of the ladder would just reach the top.

3215. (Proposed by S. Watson.)—Through two points taken at random on the circumference of a given circle, lines are drawn in random directions: find the chance that they will intersect within the given circle.

Solution by the Rev. J. Wolstenholme, M.A.

$$\text{Chance} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \left\{ \int_0^{\phi} \frac{\theta}{2\pi} \, d\theta + \int_{\phi}^{2\pi} \frac{2\pi - \theta}{2\pi} \, d\theta \right\} = \frac{1}{3}.$$

Or, we get four random points on the circle. Now wherever these lie, there are two pairs of joining lines which intersect without, and one which intersects within, the circle. Hence, on the whole, the chance of intersecting within is $\frac{1}{3}$.

3095. (Proposed by S. Roberts, M.A.)—Given three angles formed by inclined straight lines, prove that the locus of a point such that from it can be drawn three right lines cutting off equal intercepts from corresponding arms of the angles is a conic.

Solution by the Proposer.

Taking the angles for Cartesian axes, we have

$$Ax + By + C = 0, \quad Ax' + By' + C = 0, \quad Ax'' + By'' + C = 0;$$

therefore the locus is $$\begin{vmatrix} x, & y, & 1 \\ x', & y', & 1 \\ x'', & y'', & 1 \end{vmatrix} = 0.$$

3168. (Proposed by Rev. J. White, M.A.)—It is required to make the lower side of a rocker, which is loaded equally at both ends, of such a form that, when rolling on a horizontal platform, one end will have a uniform preponderance over the other.

Solution by the Proposer.

Let a small circle be described round the common centre of gravity: the lower side of the rocker must be a portion of the involute of that circle, thus:—

The line of support will be a normal to the involute, and will therefore always pass at a uniform distance from the centre of gravity—the centre of the circle from which it is unwound.

[A rocker of this form is required in the construction of Captain Moncrieff's protected barbette gun carriage.]

3210. (Proposed by the Editor.)—At a chess tournament the players are supposed to be divisible into two classes, the odds on a member of the first in a game with a member of the second being 2 to 1. The second class is twice as numerous as the first. A player is observed to win a game, and a bet of 7 to 10 is made that he belongs to the first class. Show that there are 18 players engaged.

I. *Solution by* Hugh McColl.

There were x players, say, of the first class, and consequently $2x$ of the second. As each of the whole $3x$ players may win a victory over $3x-1$ players, the total number of ways in which a victory may be won is $3x\,(3x-1)$. For a similar reason, the number of ways in which a victory is possible when first class players are opposed to first class players only, is $x\,(x-1)$. Also, since each first class player may win a victory over each second class player, and *vice versâ*, the number of ways in which a victory may be obtained when first class players are opposed to second class players, is $4x^2$; and of these (since it is 2 to 1 on a first class player against a second class player) two-thirds, that is $\frac{8}{3}x^2$ victories, will probably be gained by first class players. Consequently, out of the whole $3x\,(3x-1)$ victories, $x\,(x-1)+\frac{8}{3}x^2$ will probably be gained by first class players. Again, since the odds are 7 to 10, in other words, since the probability is $\frac{7}{17}$, for any given victory being won by a first class player, out of the whole $3x\,(3x-1)$ victories seven-seventeenths, that is $\frac{21}{17}x\,(3x-1)$ victories, will probably be gained by first class players. Equating this with the former expression, we have

$$\tfrac{21}{17}x\,(3x-1)=x\,(x-1)+\tfrac{8}{3}x^2,$$

from which we get $3x=18$.

II. *Solution by the* Rev. J. Wolstenholme, M.A.

If $3r$ be the number engaged, the number of pairs which can be taken out of the first class is $\dfrac{r\,(r-1)}{2}$; out of the second $\dfrac{2r\,(2r-1)}{2}$; and, one out

of each, is $2r^3$. The chance that the winner belongs to the first class, is in the first case 1, in the second 0, and in the third $\frac{2}{9}$. Hence, the whole chance that the winner of a particular game belongs to the first class is

$$\frac{r-1+\frac{2}{9}\times 4r}{r-1+4r-2+4r} = \frac{11r-3}{9\,(3r-1)};$$

and, in order that r may be 6, the odds should be 63 : 153 − 63, or 7 : 17 − 7, or 7 : 10. That is, the bet ought to be 10 : 7 *against* his being one of the first class.

3099. (Proposed by J. J. WALKER, M.A.)—The rectangle under tangents drawn from any point to a central conic is to the rectangle under lines drawn to the foci as the line drawn to the middle point of the chord of contact to that drawn to the centre. Prove this, and show what is the corresponding property for the parabola.

I. *Solution by the* REV. J. WOLSTENHOLME, M.A.

If the excentric angles of two points P, Q on the ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right)$ be α, β, O the pole of PQ, S, S′ the foci, the coordinates of O are $\frac{a\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}$, $\frac{b\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}$, and we get immediately

$$\mathrm{OP}^2.\mathrm{OQ}^2 = \tan^4\tfrac{1}{2}(\alpha-\beta)\,(a^2\sin^2\alpha+b^2\cos^2\alpha)\,(a^2\sin^2\beta+b^2\cos^2\beta),$$

$$\mathrm{OS}^2.\mathrm{OS}'^2 = \sec^4\tfrac{1}{2}(\alpha-\beta)\,(a^2\sin^2\alpha+b^2\cos^2\alpha)\,(a^2\sin^2\beta+b^2\cos^2\beta)$$

whence $$\frac{\mathrm{OP}.\mathrm{OQ}}{\mathrm{OS}.\mathrm{OS}'} = \sin^2\tfrac{1}{2}(\alpha-\beta),$$

and if V be the centre of the chord PQ,

$$\frac{\mathrm{OV}}{\mathrm{OC}} = 1-\frac{\cos\alpha+\cos\beta}{\dfrac{2\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}} \equiv 1-\cos^2\tfrac{1}{2}(\alpha-\beta) \equiv \sin^2\tfrac{1}{2}(\alpha-\beta) = \frac{\mathrm{OP}.\mathrm{OQ}}{\mathrm{OS}.\mathrm{OS}'}.$$

If we write this OP . OQ : OS . OV = OS′ : OC, and then remove C and S′ to ∞, the limiting ratio of OS′ : OC is 2 : 1; or in the parabola

$$\mathrm{OP}.\mathrm{OQ} = 2\mathrm{OS}.\mathrm{OV}.$$

II. *Solution by the* PROPOSER; R. TUCKER, M.A.; *and others.*

Let P be the point (x', y') from which tangents PT, PT′ are drawn to the conic $b^2x^2+a^2y^2-a^2b^2=0$, PF, PF′ to the foci, and PC to the centre, which last line bisects the chord of contact TT′ in O. Then the circle PTT′ is (Quest. 2949)

$$(b^2x'^2+a^2y'^2)\,(x^2+y^2)-b^2(r^2+c^2)x'x-a^2(r^2-c^2)\,y'y+c^2(b^2x'^2-a^2y'^2) = 0,$$

and the circle PFF′ is $y'(x^2+y^2)-(r'^2-c^2)\,y-c^2y'=0$,

where $$c^2 = a^2-b^2,\quad r'^2 = x'^2+y'^2.$$

The chord of intersection, PQ, of these circles is

$$y'(r'^2+c^2)\,x-x'(r'^2-c^2)\,y-2c^2x'y' = 0;$$

and it is readily proved that this line makes the same angle with PF′ as PC

with PF, either by calculating the tangents of these angles, or by one of the formulæ given in Quest. 3060. Hence, PQ and PC are also equally inclined to PT′ and PT also; therefore PT . PT′ = PO . PQ, and PF . PF′ = PC . PQ: whence, &c.

Otherwise, the equation which gives the squares of the tangents may be readily formed. For, supposing T to be (x, y), PT $= r$, $\xi = x-x'$, $\eta = y-y'$, $k^4 = b^2x'^2+a^2y'^2-a^2b^2$, it remains to eliminate ξ, η between $\xi^2+\eta^2-r^2=0$, $b^2\xi^2+a^2\eta^2+2b^2x'\xi+2a^2y'\eta+k^4=0$, and $b^2\xi^2+a^2\eta^2+k^4=0$. The result is

$$(b^2x'^2+a^2y'^2)^2r^4-2k^4(k^4r'^2+b^4x'^2+a^4y'^2)\,r^2+\left\{c^4-2c^2(x'^2-y'^2)+r'^4\right\}k^8=0.$$

Now $c^4-2c^2(x'^2-y'^2)+r'^4 = \text{PF}^2 . \text{PF}'^2$, and k^4, $b^2x'^2+a^2y'^2$ are proportional to the perpendiculars let fall from P on TT′ and on a parallel to this chord through the centre; that is, proportional to PO and PC.

I may remark that k^2 is equal to the area of the quadrilateral PTCT′, though possibly the remark is not new.

3159. (Proposed by A. Martin.)—Find general integral values for x, y, z that will make $xy-1=\square$, $xz-1=\square$, and $yz-1=\square$.

I. *Solution by* Samuel Bills.

Assume $xy-1=p^2$; then $x=\dfrac{p^2+1}{y}$.

Now assume $z=x+y\pm 2p$; then, substituting this in (2) and (3), we have

$$x^2+xy\pm 2px-1 = x^2\pm 2px+p^2 = (x\pm p)^2,$$
$$yx+y^2\pm 2py-1 = y^2\pm 2py+p^2 = (y\pm p)^2;$$

so that all the conditions are satisfied.

To find integers, we have only to take y a divisor of p^2+1; p may be taken at pleasure, and each value of p will give two values of z.

Let $p=1$ and $y=1$, then $x=2$ and $z=5$ or 1;
let $p=2$ and $y=1$, then $x=5$ and $z=2$ or 10;
let $p=3$ and $y=2$, then $x=5$ and $z=1$ or 13;
let $p=7$ and $y=5$, then $x=10$ and $z=1$ or 29.

II. *Solution by* Asher B. Evans, M.A.

Let $x=a^2+b^2$, $y=c^2+d^2$, $z=e^2+f^2$; then we have

$$\left.\begin{aligned} xy-(bc-ad)^2 &= (ac+bd)^2,\\ xz-(be-af)^2 &= (ae+bf)^2,\\ yz-(de-cf)^2 &= (ce+df)^2,\end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

From (1) it is evident that it only remains to make

$$bc-ad=\pm 1,\quad be-af=\pm 1,\quad de-cf=\pm 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

Let $e=a+c$ and $f=b+d$, and conditions (2) will reduce to

$$bc-ad=\pm 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (3).$$

Put $a = b+1$ in (3); then $c = d + \dfrac{d \pm 1}{b}$.

In order that c may be integral, put $d \pm 1 = bm$; then

$$d = bm \mp 1,\quad c = am \mp 1,\quad e = a(m+1) \mp 1,\quad f = b(m+1) \mp 1;$$
$$x = a^2 + b^2 = a^2 + (a-1)^2,$$
$$y = c^2 + d^2 = (am \mp 1)^2 + (am - m \mp 1)^2,$$
$$z = e^2 + f^2 = (am + a \mp 1)^2 + (am + a - m - 1 \mp 1)^2;$$

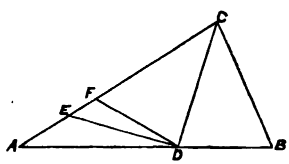
where a and m may take any integral values at pleasure.

If $a = 1$ and $m = 2$, $x = 1$, $y = 2$, $z = 5$.

if $a = 2$ and $m = 2$, $x = 5$, $y = 10$, $z = 29$.

3186. (Proposed by the Rev. R. Townsend, M.A., F.R.S.)—Prove that the perpendicular to the line bisecting the vertical angle of a triangle, drawn from the point in which the bisector meets the base, intercepts on either side a length which, measured from the vertex, is equal to the harmonic mean between the sides.

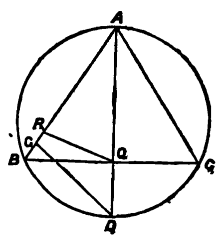
I. *Solution by the* Rev. J. White, M.A.



Let CD be the bisector of the angle ACB, and DE the perpendicular to CD. Take CF = CB, and join DF. Then by equal triangles (DBC, DFC) CD bisects the angle FDB.

In the triangle FDA the external vertical angle at D is bisected by CD, and the internal vertical angle must be bisected by ED, which is perpendicular to CD; therefore by a well known theorem the base is cut harmonically in A, E, F, and C; and CE is a harmonic mean between CA and CF (or CB).

II. *Solution by* J. McDowell, F.R.A.S.; R. Tucker, M.A.; *and others*.



Let ABC be the triangle, and let the bisector of the angle A meet the base in Q and the circumscribed circle in D. Draw DG perpendicular to AB, and QR perpendicular to AD, the points G and R being on AB.

From the similar triangles ADG, AQR, we have AR : AQ = AD : AG;

therefore AR . AG = AD . AQ = AB . AC,

and AG = $\frac{1}{2}$(AB + AC),

(McDowell's *Exercises*, No. 77); therefore AR is the harmonic mean, &c.

3114. (Proposed by G. O. HANLON.)—Find convenient formulæ of reduction for $\int x^m (a+bx^n)^p\,dx$.

Solution by the PROPOSER.

Assuming $m+1 = \text{A}$, $m+1+np = \text{B}$, and reducing successively by the usual formulæ, we find that the result

$$= \frac{x^{\text{A}}(a+bx^n)^{p+1}}{\text{A}a}\left\{1 - \frac{b}{a}\frac{\text{B}+n}{\text{A}+n}x^n + \frac{b^2}{a^2}\frac{(\text{B}+n)(\text{B}+2n)}{(\text{A}+n)(\text{A}+2n)}x^{2n}\dots\dots + \left(-\frac{b}{a}\right)^{s-1}\frac{\lfloor \text{B}+(s-1)n}{\lfloor \text{A}+(s-1)n}x^{(s-1)n}\right\}$$
$$+ \left(-\frac{b}{a}\right)^{s}\frac{\lfloor \text{B}+sn}{\lfloor \text{A}+(s-1)n}\frac{1}{\text{A}}\int x^{m+sn}(a+bx^n)^p\,dx$$

$$= \frac{x^{\text{A}-n}(a+bx^n)^{p+1}}{\text{B}b}\left\{1 - \frac{a}{b}\frac{\text{A}-n}{\text{B}-n}x^{-n} + \frac{a^2}{b^2}\frac{(\text{A}-n)(\text{A}-2n)}{(\text{B}-n)(\text{B}-2n)}x^{-2n}\dots\dots + \left(-\frac{a}{b}\right)^{s-1}\frac{\lfloor \text{A}-(s-1)n}{\lfloor \text{B}-(s-1)n}x^{-(s-1)n}\right\}$$
$$+ \left(-\frac{a}{b}\right)^{s}\frac{\lfloor \text{A}-sn}{\lfloor \text{B}-(s-1)n}\frac{1}{\text{B}}\int x^{m-sn}(a+bx^n)^p\,dx$$

$$= -x^{\text{A}}(a+bx^n)^{p+1}\left\{\frac{1}{an(p+1)} + \frac{\text{B}+n}{a^2n^2(p+1)(p+2)}(a+bx^n) + \frac{(\text{B}+n)(\text{B}+2n)}{a^3n^3\lfloor p+3}(a+bx^n)^2\dots\dots + \frac{\lfloor \text{B}+(r-1)n}{a^rn^r\lfloor p+r}(a+bx^n)^{r-1}\right\}$$
$$+ \frac{\lfloor \text{B}+rn}{a^rn^r\lfloor p+r}\int x^m(a+bx^n)^{p+r}\,dx$$

$$= \frac{px^{\text{A}}(a+bx^n)^p}{\text{B}}\left\{\frac{1}{p} + \frac{an}{\text{B}-n}(a+bx^n)^{-1} + \frac{a^2n^2(p-1)}{(\text{B}-n)(\text{B}-2n)}(a+bx^n)^{-2}\dots\dots + \frac{(an)^{r-1}\lfloor p-(r-2)}{\lfloor \text{B}-(r-1)n}(a+bx^n)^{-(r-1)}\right\}$$
$$+ \frac{(an)^r\lfloor p-(r-1)}{\lfloor \text{B}-(r-1)n}\frac{p}{\text{B}}\int x^m(a+bx^n)^{p-r}\,dx,$$

NOTE.—The above formulæ are specially useful when the integral is required between such limits as will make the first factor in the result vanish.

3131. (Proposed by MORGAN JENKINS, M.A.)—In a rectangle of mn squares, divided regularly into n rows and m columns, p black squares are placed in a specified manner: find the probability that a rook, placed on any square at random, may move without changing colour; also the minimum and maximum value of the probability for different arrangements of the p black squares.

Solution by the PROPOSER.

Let $x_1, x_2, \ldots\ldots x_n$ be the numbers of black squares in the respective rows, and $y_1, y_2, \ldots\ldots y_m$ " " " " columns; ϖ the probability required, or $(1-\varpi)$ the probability of changing colour.

Then, if the rook be placed on any of the x_1 black squares in a row, there are $(m-x_1)$ white squares in the remainder of the row; and if on any of the $(m-x_1)$ white squares, x_1 black squares. The total number of moves in that row is $m(m-1)$, of which $2x_1(m-x_1)$ are with change of colour. Hence, taking into account all the rows, and in like manner all the columns,

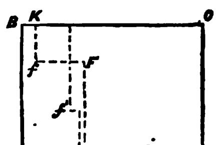
$$\frac{1-\varpi}{2} = \frac{\Sigma x(m-x)+\Sigma y(n-y)}{n\{m(m-1)\}+m\{n(n-1)\}}.$$

Let y, y' be the numbers of black squares in two columns respectively; then if $y>y'$ we must have in the y column at least one black square accompanied, in the same row, by a white square in the y' column; and if $y-y'$ be also >1, but not otherwise, the interchange of the black squares will reduce the value of $y-y'$, leaving the values of $y+y'$, of the other y's and of the x's unaltered. Since

$$y(n-y)+y(n-y') = n(y+y')-\tfrac{1}{2}\{(y+y')^2+(y-y')^2\},$$

this interchange will increase $\tfrac{1}{2}(1-\varpi)$, or diminish ϖ.

By continuing this process of lateral interchange we can at last, without altering the x's, make every pair of y's differ by not more than 1; and then again, without altering the y's, we can also make every pair of x's differ by not more than 1; and then ϖ is a minimum.



To find the maximum value of ϖ, we must distinguish between the values of m and n. Let n be not $<m$. Then I will show that the maximum value of ϖ will be obtained by filling with black squares as many as possible of the columns (whose length is n), and placing the remaining black squares in a single column.

Since any two rows, or columns, may be interchanged without altering the probability in question, we may suppose the rows and columns to be arranged in any given manner. Then, for convenience, assigning an uniform weight to each black square and no weight to the white squares, suppose the rectangle OAPB placed with the columns vertical and in descending order of weight from right (OA) to left (BP), and the rows in descending order from top (OB) to bottom (AP).

Then if a white square were to the right hand of a black in the same row, by interchanging the black and the white we should increase ϖ. For if y, y' be the weights of the columns containing the white and black squares respectively, y is $>y'$, and as before ϖ was decreased by diminish-

ing $y-y'$, so now by increasing $y-y'$, ϖ would be increased. Thus, we may suppose that the black squares lie wholly to the right hand of the rows, and in like manner at the top end of the columns, so that the black squares are separated from the white by a continuous zigzag boundary line.

In this kind of arrangement we see that, if u, v be the coordinates, measured from OA and OB, of a black square,

$$\Sigma(u+v) = \Sigma\left\{\tfrac{1}{2}x(x+1)+\tfrac{1}{2}y(y+1)\right\} = \left\{\tfrac{1}{2}\Sigma(x^2+y^2)\right\}+p,$$

since $$\Sigma x = \Sigma y = p \text{ .... } (2).$$

From the general value of $\frac{1}{2}(1-\varpi)$, ϖ is a maximum when $\Sigma(x^2+y^2)$ is so. Hence, if we call those diagonals which are nearest in direction to AB direct diagonals, we may move the black squares in their own direct diagonals without alteration of probability, or move them into direct diagonals further away from O with a gain of probability; since in the former case $\Sigma(u+v)$ is obviously unaltered, in the latter is increased. But in each case those changes only must be considered which ultimately produce the same kind of arrangement as before; viz., one in which the black squares are closely packed, and in descending order of weight, in the same directions as before; for in any other arrangement equation (2) does not hold. Now if the black squares be pushed close in their own direct diagonals towards A, the arrangement of rows and columns will be as required. For if we examine the boundary line $CcDd\ldots K$ measured from OA to OB, where C and K may coincide with A and B respectively, the abutting white squares $C_1D_1\ldots K$ are one more in number than the abutting black squares $c_1d_1\ldots f$; and as a direct diagonal travels towards O, the gain in the increment of its weight is 1 every time it crosses an abutting black square, and the loss in that increment 1 every time the diagonal crosses an abutting white square; and it cannot cross an abutting white square (E) without having previously crossed an abutting black square (e). It follows that the weight of the direct diagonal cannot begin to decrease until it has crossed the abutting white square which is nearest to the direct diagonal through O, and it then continues quite full of black squares; nor can the weight of the direct diagonal begin to be stationary until it has crossed the abutting white square nearest but one to the direct diagonal through O, nor therefore until it has crossed both C and K.

These considerations are sufficient to show that the new arrangement of the rows and columns will be of the required nature; and, as has been shown, the value of ϖ is now the same as before. The width of the horizontal portions of the boundary cannot now be greater than that of one square, and hence abutting black squares can be continually moved into abutting white squares below them with a gain in probability until the black squares occupy the above-stated position.

To make use of the results obtained, let ϖ_μ, ϖ_M denote the minimum and maximum values of ϖ respectively; and, adopting Mr. Whitworth's notation, let $\lfloor\frac{a}{b}\rfloor$ denote the integer next not-greater than $\frac{a}{b}$.

Then for ϖ_μ we have $p-n\,.\lfloor\frac{p}{n}\rfloor$ rows of weight $\lfloor\frac{p}{n}\rfloor+1$,

and $n-p+n\,.\lfloor\frac{p}{n}\rfloor$ " " $\lfloor\frac{p}{n}\rfloor$,

and symmetrically for the columns.

Substituting in (1), and reducing, we have $\frac{1}{2}(1-\varpi_\mu) =$

$$\frac{p\left\{m+n-2\left[\left\lfloor\frac{p}{m}\right\rceil+\left\lfloor\frac{p}{n}\right\rceil\right]-2\right\}+m.\left\lfloor\frac{p}{m}\right\rceil\left[\left\lfloor\frac{p}{m}\right\rceil+1\right]+n.\left\lfloor\frac{p}{n}\right\rceil\left[\left\lfloor\frac{p}{n}\right\rceil+1\right]}{mn\,(m+n-2)}$$

For ϖ_M we have the numbers and weights of the rows as before, but

$\left\lfloor\frac{p}{n}\right\rceil$ columns of weight n, and 1 column of weight $p-n.\left\lfloor\frac{p}{n}\right\rceil$.

Substituting in (1), and reducing,

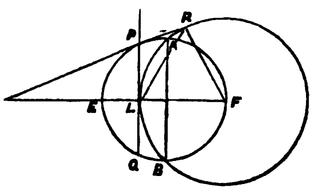
$$\frac{1-\varpi_M}{2}=\frac{-p^2+p\left\{m+(n-1)\left[2\left\lfloor\frac{p}{n}\right\rceil+1\right]\right\}-n\,(n-1)\left\lfloor\frac{p}{n}\right\rceil\left[\left\lfloor\frac{p}{n}\right\rceil+1\right]}{mn\,(m+n-2)}$$

In both these results we shall find that an equivalent value is obtained by substituting $mn-p$ for p, paying attention to the distinction between the cases when $\frac{p}{m}$ or $\frac{p}{n}$, or both, are integers.

3153. (Proposed by the Rev. A. F. Torry, M.A.)—A conic circumscribes a quadrilateral ABCD; E is the pole of AB, F that of CD; and AB, CD intersect in O. Through E is drawn a straight line, cutting the conic in Q, R, and CD in M; and upon this line a point P is taken to make the range (QPRM) harmonic. Show that the locus of P will be a conic AEBCFD, the tangents to which at E, F pass through O; and that the tangents from O to the first conic will pass through the four points in which the common tangents to the two conics touch the second.

Solution by the Rev. J. Wolstenholme, M.A.

Project two of the points A, B, C, D (say C, D) into the circular points at infinity, the conic becomes a circle through A, B; E is the pole of A, B, F the centre of the circle, and O the point at infinity on AB. Any straight line drawn through E, cutting the conic in Q, R and CD in M, will be divided harmonically in P, the middle point of QR, and the locus of P is



obviously the circle on E, F as diameter; that is, the conic ABCDEF; and the tangents to it at E, F, being perpendicular to EF, will pass through O. Also the tangents from O to the first circle will pass through the points when the common tangents touch the second circle; for if one of these tangents to the first circle at L meet the second circle in P, Q, and PR be drawn touching the first,

$$180^\circ - \angle\text{RPL} = \angle\text{RFL} = 2\angle\text{PFL} = \angle\text{PFQ} = 180^\circ - \angle\text{PEQ};$$

therefore RP is a tangent at P to the second circle.

One of the tangents through O to the first circle will not meet the other

circle in real points, but an analytical proof which would prove it for one tangent would do it also for the other.

The last part was set in the Senate-House Examination in the Three Days' Problem Paper of 1868, and was then obtained as a particular case of the following, which has appeared in the *Educational Times*:—"A circle is drawn through the foci of an ellipse: the tangents to the ellipse at the ends of the minor axis will pass through the points of contact with the circle of common tangents to the ellipse and circle."

3179. (Proposed by R. W. GENESE.)—Given the perimeter and an angle of a triangle, prove that the area is a maximum when the sides containing the given angle are equal.

I. *Solution by* R. TUCKER, M.A.; Rev. J. WOLSTENHOLME, M.A.; *and others.*

Measure off on the sides of the given angle, from the vertex, distances equal to the semiperimeter; then the third sides of all the triangles will envelope the circle touching the given sides at the points determined above.

Now we can easily see that the greatest circle that can be drawn touching the sides of the given angle so as not to cut the above circle, is that which *touches* the same circle; and the area of the triangle ($=rs$) is greatest when r is greatest—that is, in this case, when the third side touches the envelope on the bisector of the given angle, or when the triangle is isosceles.

II. *Solution by* STEPHEN WATSON; A. MARTIN; *and others.*

Let the perimeter $= s$, the given angle $= \alpha$, and the sides containing $\alpha = x$ and y. Then

$$\tfrac{1}{2}xy\sin\alpha = \text{max.}; \quad \text{or,} \quad xy = \text{max} \qquad (1),$$

subject to the condition $x^2+y^2-2xy\cos\alpha = (s-x-y)^2$;

or $$2s(x+y)-s^2 = 4xy\cos^2\tfrac{1}{2}\alpha = \text{max} \qquad (2).$$

From (1) and (2), $\quad ydx+xdy = 0, \quad dx+dy = 0$;

therefore $x=y$, and the maximum area is easily found to be $\dfrac{s^2\sin\alpha}{8(1+\sin^2\frac{1}{2}\alpha)^2}$

3118. (Proposed by R. TUCKER, M.A.)—If, in a spherical triangle, $A = \frac{2}{5}\pi$, $B = \frac{1}{5}\pi$, $C = \frac{1}{2}\pi$, prove that (1) $\tan c = 2$, $a+b = \frac{1}{2}\pi$, $c = 2b$; (2) the median arc from C to AB is $\frac{1}{5}\pi$, and the perpendicular arc is $\frac{1}{6}\pi$; (3) $\tan\frac{1}{2}a\tan\frac{1}{2}b = \tan\frac{1}{20}\pi$; (4) $\tan r\tan r_1\tan r_2\tan r_3 = \frac{1}{20}$.

Solution by the PROPOSER.

1. We have

$$\cos c = \cot A \cot B = \frac{\sqrt{5}-1}{4} \cdot \frac{\sqrt{5}+1}{4} \cdot \frac{1}{2 \sin 18^\circ \cos^2 18^\circ} = \frac{1}{\sqrt{5}};$$

therefore $\tan c = 2, \quad \cos a = \dfrac{\cos A}{\sin B} = \dfrac{1}{2 \cos 18^\circ};$

therefore $\dfrac{1}{5} = \cos^2 c = \cos^2 a \cos^2 b = \dfrac{\cos^2 b}{4 \cos^2 18^\circ} = \dfrac{2 \cos^2 b}{\sqrt{5}(\sqrt{5}+1)};$

whence $\cos^2 b = \dfrac{\sqrt{5}+1}{2\sqrt{5}}, \quad$ and $\quad \sin^2 b = \dfrac{\sqrt{5}-1}{2\sqrt{5}},$

or $2 \sin^2 b = 1 - \dfrac{1}{\sqrt{5}} = 1 - \cos c, \quad$ and $\quad \cos 2b = \cos c;$

therefore $c = 2b, \quad$ or $\quad = 2\pi - 2b,$

$$\cos b = \frac{\cos B}{\sin A} = \frac{\cos 36^\circ}{\cos 18^\circ} = 2 \cos a \cos 36^\circ;$$

therefore $\dfrac{\cos b}{2 \cos a} = 2 \cos^2 18^\circ - 1 = \dfrac{1}{2 \cos^2 a} - 1,$

or $\cos c = \cos a \cos b = 1 - 2 \cos^2 a = -\cos 2a;$

therefore $2a + c = \pi;$

hence we obtain (the only possible case) $a + b = \frac{1}{2}\pi$.

2. If δ be the length of the median arc, then (Todhunter, p. 44, Ex. 8)

$$2 \sin^2 \delta = \frac{1}{2 \cos^2 \frac{1}{2}c} = \frac{1}{1 + \cos c} = \frac{\sqrt{5}(\sqrt{5}-1)}{4};$$

therefore $\cos 2\delta = \frac{1}{4}(\sqrt{5}-1) = \cos \frac{2}{5}\pi, \quad$ therefore $\quad \delta = \frac{1}{5}\pi$.

Similarly (p. 44, Ex. 12), if δ' be the perpendicular arc,

$$\cos \delta' = \tfrac{1}{2}\sqrt{5}\,(1 - 2\cos^2 c)^{\frac{1}{2}} = \tfrac{1}{2}\sqrt{5} \cdot \sqrt{\tfrac{3}{5}} = \tfrac{1}{2}\sqrt{3} = \cos \tfrac{1}{6}\pi;$$

therefore $\delta' = \frac{1}{6}\pi$.

3. $\operatorname{Tan} R = \dfrac{\tan \frac{1}{2}c}{\cos 9^\circ}, \quad \tan R_1 = \dfrac{\tan \frac{1}{2}a}{\sin 9^\circ}, \quad \tan R_2 = \dfrac{\tan \frac{1}{2}b}{\sin 9^\circ}, \quad \tan R_3 = \dfrac{\tan \frac{1}{2}c}{\sin 9^\circ}.$

But (p. 66, Ex. 5) $\tan R_1 \tan R_2 \tan R_2 = \tan R \sec^2 S;$

therefore $\dfrac{\tan \frac{1}{2}a \tan \frac{1}{2}b \tan \frac{1}{2}c}{\sin^3 9^\circ} = \dfrac{\tan \frac{1}{2}c}{\cos 9^\circ} \operatorname{cosec}^2 9^\circ;$

therefore $\tan \frac{1}{2}a \tan \frac{1}{2}b = \tan \frac{1}{20}\pi.$

4. $\operatorname{Tan} r = \dfrac{\sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A} \sin a = \dfrac{3-\sqrt{5}}{2\sqrt{2}} \sin a, \quad \tan r_1 = \dfrac{\sqrt{2}}{\sqrt{5}+1} \tan a,$

$$\tan r_2 = \frac{\sqrt{5}+1}{2\sqrt{2}} \cos^2 a, \quad \tan r_3 = \frac{\sqrt{5}+1}{2\sqrt{2}} \sin a;$$

therefore $\tan r \tan r_1 \tan r_2 \tan r_3 = \dfrac{\sqrt{5}-1}{8} \sin^2 a \cos^2 a \tan a$

$$= \frac{\sqrt{5}-1}{8} \cos^2 c \frac{\cos b}{\cos a} = \frac{\sqrt{5}-1}{8} \cdot \frac{1}{5} \cdot \frac{2(\sqrt{5}+1)}{4} = \frac{1}{20}.$$

3224. (Proposed by R. W. Genese.)—From any point T on a straight line OX, TQ is drawn making a constant angle with the polar of T with respect to a fixed conic; show that the envelope of TQ is in general a parabola which reduces to a point if the constant angle be that which OX makes with the direction conjugate to OX.

3244. (Proposed by the Rev. A. F. Torry, M.A.)—Through any point P, within or without a parabola whose focus is S, a double ordinate QQ′ is drawn; the polar of P cuts the axis in M; the perpendicular from P upon this polar meets it in N and the axis in R: show that M, N, Q, R, Q′ all lie on a circle whose centre is S.

Solution by Dr. Hirst, F.R.S.

The polars t of the several points T of the range whose axis is OX form a pencil homographic with that range, and having for its centre P the pole of OX. Conceive this pencil to be turned round P, in a definite sense, through the constant angle, and let PQ be the new position of t, Q being on the line at infinity. It is obvious that T and Q will be corresponding points of two homographic ranges, and consequently that TQ will envelope a parabola of which OX is a tangent.

The points of contact of this parabola with OX and with the line at infinity are, by a well known theorem, the points which correspond, on each range, to the infinitely distant intersection X of their two axes. Thus, if PO, the polar of X (in other words, the diameter conjugate to the direction of OX), be turned around P, in the primitive sense, through the constant angle, it will indicate the direction of the axis of the parabola; and if PX be turned around P, in the *opposite sense*, through the same angle, its pole will be the point of contact of OX and the parabola.

Now, if the first turning were such as to bring PO into coincidence with PX, by the second PX would be brought back to PO, and it is obvious that two corresponding points of the two ranges would coincide in X. In this case, therefore, the ranges would be in perspective, that is to say, the connectors T, Q would all pass through a point R, and the parabola would reduce itself to what Professor Cayley has termed a *line-pair-point*. This point R is easily constructed by means of two positions of TQ.

The following are immediate consequences of the theorem last established.

A definite mode of rotation being understood, and α being the angle between a given straight line OX and that diameter of a conic which is conjugate to its direction, the locus of the foot of an oblique drawn from any point of OX to the polar of that point, and making an angle α therewith, is the circumference of a circle.

The perpendiculars let fall from the several points of a right line, parallel to either axis of a conic, upon the respective polars of those points, are concurrent on the other axis, and the locus of their feet is a circle.

The theorem in Question 3244 is obviously a special case of this.

3212. (Proposed by J. F. Moulton, M.A.)—A cube just filled with homogeneous fluid is made to rotate about an axis through its centre of gravity. Show that the whole pressure on its surface is independent of

what axis be chosen, and bears to the pressure on a sphere of the same surface filled with similar fluid and rotating with the same velocity the ratio $10\pi : 36$.

Solution by the Rev. R. Townsend, M.A., F.R.S.

The first part of the property is true, not only for a hollow cube, but for any closed vessel the area of whose inner surface exposed to the action of the fluid has, for some internal point, its three principal moments of inertia equal, with respect to any axis of rotation passing through that point. For, denoting by A, B, C the three principal moments of inertia of the inner area S of any closed vessel, filled with homogeneous fluid, with respect to any internal point O, by I its moment of inertia with respect to any axis of rotation passing through O, by α, β, γ the three direction angles of the axis of rotation, by p the perpendicular distance of any element dS of S from the axis, by ρ and ω the density and angular velocity of the fluid, and by P the entire pressure on S; then, since evidently,

$$P = \tfrac{1}{2}\rho\omega^2 \iint p^2 dS = \tfrac{1}{2}\rho\omega^2 \, I,$$

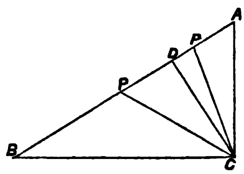
therefore, &c.; I, as is well known, being independent of α, β, γ, when, as in the case of the hollow cube for its centre, $A = B = C$.

That a point O should exist within S for which $A = B = C$, it is, as is well known, necessary that for its centre of gravity G two of its three principal moments of inertia must be equal, and each of them less, or at least not greater, than the third; when these conditions are fulfilled, the two foci O and O′ of its central ellipsoid of inertia, which is then evidently a prolate spheroid, possess each the required property.

By comparing the well known values of $A = B = C$ for the surfaces of a cube and sphere of equal area, the second part of the proposed question is evident.

3204. (Proposed by A. Martin.)—Find the average distance of the right angle of a given right angled triangle from the hypotenuse.

Solution by the Proposer.



In the right-angled triangle ABC, draw CD perpendicular to the hypotenuse AB; and let x = the distance of any point P in the hypotenuse from D, and M = the mean distance required; then $BP = \left(x^2 + \frac{a^2b^2}{c^2}\right)^{\frac{1}{2}}$.

The limits of x are 0 and $\frac{a^2}{c}$ when P is in CD, and 0 and $\frac{b^2}{c}$ when P is in AD; hence, integrating between these limits, we have

$$M = \frac{1}{c}\int\left(x^2 + \frac{a^2b^2}{c^2}\right)^{\frac{1}{2}} dx = \frac{a^3 + b^3}{2c^2} + \frac{a^2b^2}{2c^3}\log\left(\frac{ab\,(a+c)\,(b+c)}{c^4}\right).$$

[With Mr. MARTIN's hypothesis, viz., that the points P are *equably distributed over* AB, the average of the *square* of CP is

$$\frac{1}{c}\int\left(x^2+\frac{a^2b^2}{c^2}\right)dx=\frac{a^6+b^6}{3c^4}+\frac{a^2b^2}{c^2}=\frac{a^6+3a^2b^2c^2+b^6}{3c^4},$$

which reduces to the simple value $\frac{1}{3}c^2$. Thus the average area S_1 of the circle on CP as diameter is one-third of the circle drawn round the triangle ABC. If the lines are drawn *at equal angular intervals* ($\theta=\text{PCD}$), the average area S_2 of the circle on CP is

$$S_2=\frac{1}{\frac{1}{2}\pi}\int\tfrac{1}{4}\pi\,.\,\text{CD}^2\sec^2\theta\,d\theta=\tfrac{1}{2}\text{CD}^2(\tan\text{ACD}+\tan\text{BCD})$$
$$=\tfrac{1}{2}\text{CD}\,.\,\text{AB}=\triangle\text{ABC};$$

and we have $S_1:S_2=\pi:3\sin 2A$. It is evident, *à priori*, that S_1 must be greater than S_2; since, in the former case, the points P are *uniformly* distributed over AB, and in the latter their *density decreases* as the length of CP *increases*.]

3090. (Proposed by the Rev. E. HILL, M.A.)—A portion of the water in a pond, in the shape of a vertical cylinder, radius a, is rotating about its axis, the angular velocity at any point being $k(a^n-r^n)^{\frac{1}{2}}$, where r is the distance from the axis. Find the form of the free surface, and show that the volume of the depression below the surface of the pond is $\dfrac{\pi k^2na^4}{4g(n+4)}$.

Solution by JAMES DALE.

Taking the surface of the cylinder when at rest for the plane xy, and its axis for the axis of z measured downwards, we have the forces acting along the axes $k^2(a^n-r^n)x$, $k^2(a^n-r^n)y$, g; and substituting these values in the general equation of equilibrium $Xdx+Ydy+Zdz=0$, we get

$$k^2a^n(x\,dx+y\,dy)-k^2(x^2+y^2)^{\frac{1}{2}n}(x\,dx+y\,dy)+g\,dz=0;$$

and, integrating, $k^2a^n(x^2+y^2)-k^2\dfrac{2}{n+2}(x^2+y^2)^{\frac{1}{2}(n+2)}+2gz=C;$

when $x^2+y^2=a^2$, then $z=0$, and $C=k^2a^{n+2}\dfrac{n}{n+2}$, so that the equation to the free surface is

$$k^2(x^2+y^2)\left\{a^n-\frac{2}{n+2}(x^2+y^2)^{\frac{1}{2}n}\right\}+2gz=\frac{n}{n+2}k^2a^{n+2};$$

when $x^2+y^2=0$ then $z=\dfrac{n}{2g(n+2)}k^2a^{n+2}=$ depth of the depression.

The volume of the depressed portion $=\pi\int r^2dz$, which, by the equation to the surface, $=\dfrac{\pi k^2}{2g}\displaystyle\int\left\{(x^2+y^2)^{\frac{1}{2}(n+2)}-a^n(x^2+y^2)\right\}d(x^2+y^2).$

Taking the integral between the limits $x^2+y^2=a^2$ and $x^2+y^2=0$, we have

$$\text{volume}=\frac{n}{n+4}\,\frac{\pi k^2a^4}{4g}.$$

2988. (Proposed by the Rev. A. F. TORRY, M.A.)—A, B, C, D throw with three dice in succession for a prize, the highest throw winning, and equal throws continuing the trial. At the first throw A throws 13. Find his chance of winning.

Solution by F. D. THOMSON, M.A.

The number of ways in which a particular throw r may be made when the first die counts s is equal to the number of ways in which $(r+1)$ can be thrown when the first die counts $(s+1)$.

Hence we get the following table, where the top row of figures gives the throw, and the column under each the total number of ways in which the throw can be made:—

	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
The first die 1 ...	1	2	3	4	5	6	5	4	3	2	1					
„ 2 ...		1	2	3	4	5	6	5	4	3	2	1				
„ 3 ...			1	2	3	4	5	6	5	4	3	2	1			
&c.							&	c.								
Total ...	1	3	6	10	15	21	25	27	27	25	21	15	10	6	3	1

Hence the total number of ways in which a throw *less than* 13 can be made is 160.

But if B throws 13, he throws again, and so on until he throws some number different from 13; hence the *total* number of possible ways which need be considered is 6^3-21 or 195.

Hence B's chance of failing $=\dfrac{160}{195}=\dfrac{32}{39}$.

Hence the chance that B, C, D may all fail is $\left(\dfrac{32}{39}\right)^3=\cdot552$, or A's chance of winning is $\cdot552$.

This is on the supposition that by "equal throws" is meant a throw of 13 by B, C, or D. If the question mean that, if C or D make any throw the same as B or each other, they throw again, it becomes excessively complicated.

3115. (Proposed by Dr. D. S. HART.)—To find five biquadrate numbers whose sum is a biquadrate number.

Solution by the PROPOSER.

The sum of n consecutive biquadrates,

$$1^4,\ 2^4,\ 3^4,\ 4^4,\ 5^4,\ \dots n^4, \text{ is } \tfrac15 n^5+\tfrac12 n^4+\tfrac13 n^3-\tfrac1{30}n \ \dots\dots\dots\dots\dots (1).$$

Next assume
$$(s+m)^4-s^4 \ \dots\dots\dots\dots\dots\dots\dots\dots (2).$$

If the difference of (1) and (2) be divided into biquadrate numbers (which is possible in many cases), and subtracted from (1), and then s^4 added, we shall have a set of biquadrate numbers whose sum is a biquadrate.

The rule may be stated in general terms as follows:

$$\tfrac{1}{5}n^5+\tfrac{1}{2}n^4+\tfrac{1}{3}n^3-\tfrac{1}{30}n-(\tfrac{1}{5}n^5+\tfrac{1}{2}n^4+\tfrac{1}{3}n^3-\tfrac{1}{30}n)+\left\{(s+m)^4-s^4\right\}+s^4=(s+m)^4.$$

It is evident that s must be taken greater than n, for otherwise s^4 may be included in the series of biquadrates of which (1) is composed. I have also found that n must not be assumed less than 9; also m may be taken $= 1, 2, 3, 4$, &c.

If $n=9$, $s=14$, $m=1$, then (1) is 15333, and (2) is 12209; their difference is $3124 = 1^4+2^4+3^4+5^4+7^4$; and when this is taken from the series in (1), and 14^4 added, we have $4^4+6^4+8^4+9^4+14^4 = 15^4$.

If n be taken successively $= 10, 11, 12$, &c., *ad infinitum*, s and m as before, we shall have the same numbers. This is a singular fact.

By taking $n=20$, $s=30$, and $m=4$, I find the fifteen biquadrates, $1^4, 3^4, 4^4, 5^4, 9^4, 10^4, 11^4, 12^4, 14^4, 15^4, 16^4, 17^4, 18^4, 19^4, 30^4$, whose sum $= 34^4$.

I suppose that three biquadrates, and four biquadrates, can be found, but n, s must be taken much larger numbers.

2968. (Proposed by R. Tucker, M.A.)—In Question 2947, show (1) that the point of intersection of the normals at B, C, D lies on the normal at the extremity of the diameter through A; and (2) that the radical axes of the osculating circles at B, C, D are tangents to the curve $\left(\dfrac{x}{2a}\right)^{\frac{2}{3}}+\left(\dfrac{y}{2b}\right)^{\frac{2}{3}}=1.$

Solution by the Rev. J. Wolstenholme, M.A.

1. The normals at B, C, D being each perpendicular to the opposite side of the triangles BCD, of course meet in a point; and if $\alpha, \beta, \gamma, \delta$ be the eccentric angles of A, B, C, D, $\gamma=\beta+\frac{2}{3}\pi$, $\delta=\beta+\frac{4}{3}\pi$, $\alpha=-3\beta$, therefore $\alpha+\beta+\gamma+\delta=2\pi$. But if A′ be the fourth point to which a normal can be drawn from the intersection of the normals at B, C, D, and α' its eccentric angle, $\alpha'+\beta+\gamma+\delta=$ an odd multiple of π. Hence the eccentric angles of A, A′ differ by an odd multiple of π, or A, A′ are at opposite ends of a diameter.

2. The equation of the osculating circle at θ is

$$x^2+y^2-\frac{2x}{a}(a^2-b^2)\cos^3\theta+\frac{2y}{b}(a^2-b^2)\sin^3\theta$$
$$+a^2(\cos^2\theta-2\sin^2\theta)+b^2(\sin^2\theta-2\cos^2\theta)=0,$$

and the equation of the radical axis of the osculating circle at (θ,ϕ) is

$$\frac{2x}{a}(\cos^3\theta-\cos^3\phi)-\frac{2y}{b}(\sin^3-\sin^3\phi)=3(\cos^2\theta-\cos^2\phi),$$

or $$\frac{x}{a}[3(\cos\theta-\cos\phi)+4(\cos3\theta-\cos3\phi)]-\frac{2y}{b}[3(\sin\theta-\sin\phi)$$
$$-4(\sin3\theta-\sin3\phi)]=3(\cos2\theta-\cos2\phi).$$

If these circles meet on the ellipse, we may take $\theta = \alpha - \frac{1}{3}\pi$, $\phi = \alpha + \frac{1}{3}\pi$, and

$$\left.\begin{aligned} &\text{the radical axis becomes} \quad \frac{x}{a}\sin\alpha + \frac{y}{b}\cos\alpha = \sin 2\alpha \\ &\text{and for the envelope} \quad \frac{x}{a}\cos\alpha - \frac{y}{b}\sin\alpha = 2\cos 2\alpha \end{aligned}\right\};$$

therefore $\qquad \dfrac{x}{a} = 2\cos^3\alpha, \quad \dfrac{y}{b} = 2\sin^3\alpha;$

hence the equation of the envelope is $\left(\dfrac{x}{2a}\right)^{\frac{2}{3}} + \left(\dfrac{y}{2b}\right)^{\frac{2}{3}} = 1.$

[If from any centre of curvature O be drawn two other normals P, Q, it will be found that the equation of PQ is $\dfrac{x}{a}\sin\alpha + \dfrac{y}{b}\cos\alpha = -\dfrac{1}{2}\sin 2\alpha$; or if O be the centre of curvature at D, PQ is parallel to the radical axis of the circles of curvature at B and C and at half the distance from the centre.]

2596. (Proposed by the Rev. A. F. Torry, M.A.)—Two particles are projected with the same velocity, in the same direction, at right angles to the line joining them: find the curves they will describe if they attract according to the law of nature. If the particles were equal, find what law of resistance, in the medium through which they pass, would cause them to describe parts of the same circle.

Solution by J. J. Walker, M.A.

Let m, m' be the masses of the particles, μ the absolute force, A, A′ the initial positions of m, m'. Take AO : OA′ = m' : m; and, drawing OB at right angles to OA, let x, y be the coordinates of m, at the end of the time t, referred to OA and OB as axes; x', y' those of m' referred to OA′ and OB. Let a represent the distance AA′, and u the velocity of projection of the particles, from their positions A, A′, parallel to OB. Then the line mm' will be constantly parallel to AA′, and

$$\frac{d^2x}{dt^2} = -\frac{\mu m'}{(x+x')^2}, \quad \frac{d^2x'}{dt^2} = -\frac{\mu m}{(x+x')^2}, \quad \text{whence} \quad m\frac{d^2x}{dt^2} - m'\frac{d^2x'}{dt^2} = 0;$$

and integrating, $\qquad m\dfrac{dx}{dt} - m'\dfrac{dx'}{dt} = \text{const.} = 0,$

since at A and A′ the velocities parallel to A, A′ are both zero. Integrating again, $mx - m'x' = \text{const.} = 0$, since $m\text{AO} - m'\text{OA}' = 0$. Substituting for x' in the first equation above, $\dfrac{d^2x}{dt^2} = -\dfrac{\mu m'^3}{M^2x^2}$, if $m + m' = M$. Multiplying by $2\dfrac{dx}{dt}$, and integrating, $\dfrac{dx^2}{dt^2} = \dfrac{2\mu m'^3}{M^2x} + \text{const.}$ At A this becomes

$0 = \frac{2\mu m'^2}{Ma} + \text{const.}$, whence

$$\frac{dx^2}{dt^2} = \frac{2\mu m'^2}{Ma}\,\frac{m'a - Mx}{x} \quad\text{and}\quad dt = \left(\frac{Ma}{2\mu m'^2}\right)^{\frac{1}{2}}\left(\frac{x}{m'a - Mx}\right)^{\frac{1}{2}} dx,$$

the integral of which is

$$t = \frac{a^{\frac{3}{2}}}{2\,(2\mu)^{\frac{1}{2}}\,M}\sin^{-1}\frac{2Mx - m'a}{m'a} - \frac{a^{\frac{1}{2}}}{m'\,(2\mu M)^{\frac{1}{2}}}\,(m'ax - Mx^2)^{\frac{1}{2}} + \text{const.}$$

The velocity of each particle parallel to the axis of y is constant, and equal to u, whence $y = ut$; and determining the constant from the simultaneous values $t = 0$, $Mx = m'a$, the locus of x is

$$\frac{(2\mu)^{\frac{1}{2}}Mm'}{u}\,y + \left(\frac{m'a}{M} - x^2\right)^{\frac{1}{2}} - \frac{a}{2m'}\cos^{-1}\frac{2Mx - m'a}{m'a} = 0.$$

Supposing now the particles to be equal, to find the law of resistance in the medium in which they move so that they may describe parts of the same circle, it is plain that $x = x'$; and if the forces be resolved tangentially and normally, the general equations (R being the resistance)

$$v\frac{dv}{ds} = T - R, \quad \frac{v^2}{\rho} = N \quad\text{become}\quad v\frac{dv}{ds} = -\frac{\mu m}{4x^2}\frac{dx}{ds} - R \;\ldots\ldots\; (1),$$

and $$\frac{v^2}{a} = \frac{\mu m}{4x^2}\frac{dy}{ds} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

a being the radius of the circle described.

Now $\frac{dy}{ds} = \frac{x}{a}$, $\frac{dx}{ds} = -\frac{y}{a}$, whence (1) and (2) become

$$v\frac{dv}{ds} = \frac{\mu m y}{4ax^2} - R, \quad v^2 = \frac{\mu m}{4ax};$$

from the latter of which $v\frac{dv}{ds} = -\frac{\mu m}{8x^2}\frac{dx}{ds} = \frac{\mu m y}{8ax^2}$.

Substituting in the former, $R = \frac{\mu m}{8a}\frac{y}{x^2}$.

If the resistance be supposed to vary as the density of the medium and square of the velocity, we shall have

$$\text{density} \propto \frac{R}{v^2} \propto \frac{y}{x}.$$

2981. (Proposed by the Rev. G. H. Hopkins.)—If the sum of the squares of two consecutive integers be equal to the square of another integer, find their general values, and show how to find any number of particular solutions.

Solution by Artemas Martin.

Let $\frac{1}{2}(x-1)$, $\frac{1}{2}(x+1)$, and y denote such numbers. Then

$$\tfrac{1}{4}(x-1)^2 + \tfrac{1}{4}(x+1)^2 = y^2;$$

whence $\qquad x^2-2y^2=-1$, and $x=\sqrt{(2y^2-1)}$ ....(1).

When x and y are very large numbers, the 1 under the radical may be neglected without sensible error, and we have for the superior limit of their ratio $\frac{x}{y}=\sqrt{2}$; the values of x and y are, therefore, the numerators and denominators of the odd convergents to the square root of 2 expanded as a continued fraction.

Now we have $\quad \sqrt{2}=1+\frac{1}{2+}\;\frac{1}{2+}\;\frac{1}{2+\text{\&c.}}$;

and the odd convergents are

$$\frac{1}{1},\ \frac{7}{5},\ \frac{41}{29},\ \frac{239}{169},\ \frac{1393}{985},\ \text{\&c.}$$

Writing $\frac{x}{y}$, $\frac{x_1}{y_1}$, $\frac{x_2}{y_2}$, $\frac{x_3}{y_3}$, $\frac{x_4}{y_4}$, &c.; for the above fractions, we have

$$\left.\begin{aligned}x_2&=6x_1-x\\ y_2&=6y_1-y\end{aligned}\right\},\quad \left.\begin{aligned}x_3&=6x_2-x_1\\ y_3&=6y_2-y_1\end{aligned}\right\},\ \ldots\ldots\quad \left.\begin{aligned}x_n&=6x_{n-1}-x_{n-2}\\ y_n&=6y_{n-1}-y_{n-2}\end{aligned}\right\}\ldots\ldots\ (2),$$

which afford any number of particular solutions.

And we may further deduce

$$\left.\begin{aligned}x_{2n}&=2y_n^2-\tfrac{1}{4}(y_n+y_{n-1})^2\\ y_{2n}&=\ y_n^2+\tfrac{1}{4}(y_n-y_{n-1})^2\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

which are convenient formulæ for computing large numbers.

It is obvious that the values of x, x_1, x_2, x_3, x_4, &c., and of y, y_1, y_2, y_3, y_4, &c., form each a recurring series of the second order, the "scale of relation" in each case being -1 and 6. By the method of finding the nth term of such series, we obtain

$$\left.\begin{aligned}x_n&=\frac{1}{2}\left\{(1+\sqrt{2})^{2n+1}+(1-\sqrt{2})^{2n+1}\right\}\\ y_n&=\frac{1}{2\sqrt{2}}\left\{(1+\sqrt{2})^{2n+1}-(1-\sqrt{2})^{2n+1}\right\}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ (4).$$

Hence the general values required are

$$\frac{1}{2}(x_n-1)=\frac{1}{4}\left\{(1+\sqrt{2})^{2n+1}+(1-\sqrt{2})^{2n+1}-2\right\},$$

$$\frac{1}{2}(x_n+1)=\frac{1}{4}\left\{(1+\sqrt{2})^{2n+1}+(1-\sqrt{2})^{2n+1}+2\right\},$$

$$y_n=\frac{1}{2\sqrt{2}}\left\{(1+\sqrt{2})^{2n+1}-(1-\sqrt{2})^{2n+1}\right\}.$$

Expanding the values in (4) by the "Binomial Theorem," we get

$$x_n=1+2\left(\frac{(2n+1).(2n)}{1.2}\right)+2^2\left(\frac{(2n+1)(2n)(2n-1)(2n-2)}{1.2.3.4}\right)+\ldots\ldots$$
$$\ldots\ldots+2^n(2n+1),$$

$$y_n=1+2n+2\left(\frac{(2n+1)(2n)(2n-1)}{1.2.3}\right)+\ldots\ldots+2^n.$$

Taking $n=1, 2, 3, 4, 5,$ &c., we find

$$3^2+4^2=5^2,\quad 20^2+21^2=29^2,\quad 119^2+120^2=169^2,\quad 696^2+697^2=985^2,$$
$$4058^2+4059^2=5741^2, \text{ \&c.}$$

Putting $n=5$ and $=10$ respectively in (3), we find

$$27304196^2+27304197^2=38613965^2,$$

and $\quad 1235216565974040^2+1235216565974041^2=1746860020068409^2.$

[Other Solutions are given in Vol. XII. of the *Reprint*, pp. 104—106.]

2834. (Proposed by J. J. WALKER, M.A.)—Show that the equation to the envelope referred to in Question 2780 (see *Reprint*, Vol. XI., p. 63), may be thrown into the form $U^3+V^2=0$, where $U=0$ is the equation to a similar and coaxal ellipse, $V=0$ that to its equi-conjugate diameters.

Solution by the PROPOSER.

The envelope may be very easily obtained in the form proposed by Mr. BURNSIDE's elegant method (see Solution of Question 2740, p. 42 of Vol. XI. of the *Reprint*). For the equation to the chord is

$$\xi^2-\eta^2-\frac{x}{a}\xi+\frac{y}{b}\eta=0\ \left(\text{where } \xi=\frac{x'}{a},\ \ \eta=\frac{y'}{b}\right), \text{ and } \xi^2+\eta^2-1=0.$$

Multiplying the second equation by λ, and adding to the first, we have

$$(\lambda+1)\xi^2+(\lambda-1)\eta^2-\frac{x}{a}\xi+\frac{y}{b}\eta-\lambda=0.$$

The discriminant of this, equated to zero, gives

$$\lambda^3+\frac{1}{4}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-4\right)\lambda-\frac{x^2}{4a^2}+\frac{y^2}{4b^2}=0.$$

The discriminant of this is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-4\right)^3+27\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=0,$$

which is the form proposed in the Question.

3221. (Proposed by S. ROBERTS, M.A.)—Determine the condition that a straight line and two conics may have two common points of intersection.

I. *Solution by* C. W. MERRIFIELD, F.R.S.

Let the conics be

$$U=Ax^2+By^2+Cz^2+2Lyz+2Mzx+2Nxy=0,$$
$$V=A_1x^2+\ \ldots\ldots \qquad\qquad =0.$$

Then the condition that the linear combination shall contain as one of its factors the line $ax+by+cz=0$, may be obtained by equating $U-kV=0$ term for term to $(ax+by+cz)(lx+my+nz)=0$,
and eliminating k, l, m, n. The resulting equations are

$$\begin{aligned} A-kA_1 &= al, & cm+bn &= 2(L-kL_1),\\ B-kB_1 &= bm, & an+cl &= 2(M-kM_1),\\ C-kC_1 &= cn, & bl+cn &= 2(N-kN_1). \end{aligned}$$

Substituting the values of l, m, n, obtained from the first set, in the second, we find

$$k=\frac{b^2C+c^2B-2bcL}{b^2C_1+c^2B_1-2bcL_1}=\frac{c^2A+a^2C-2caM}{c^2A_1+a^2C_1-2caM_1}=\frac{a^2B+b^2A-2abN}{a^2B_1+b^2A_1-2abN_1}.$$

Neglecting k, these are the conditions required.

II. *Solution by the* Proposer.

Let the equations of the line and conics be

$$Ax+By+Cz=0,$$
$$a\,x^2+b\,y^2+c\,z^2+2d\,yz+2exz+2fxy=0,$$
$$a'x^2+b'y^2+c'z^2+2d'yz+2e'xz+2f'xy=0.$$

Eliminating successively x, y, z, we get the set of equations

$$\left.\begin{aligned} (a\,B^2+b\,A^2-2f\,AB)\,y^2+2\,(aBC-eAB-fAC+dA^2)\,yz&\\ +\,(aC^2+cA^2-2eAC)\,z^2&=0\\ (a'B^2+b'A^2-2f'AB)\,y^2+\quad\ldots\ldots\ldots&=0 \end{aligned}\right\}\ldots(a),$$

$$\left.\begin{aligned} (a\,B^2+b\,A^2-2f\,AB)\,x^2+2\,(bAC-dAB-fBC+eB^2)\,xz&\\ +\,(bC^2+cB^2-2dBC)\,z^2&=0\\ (a'B^2+b'A^2-2f'AB)\,x^2+\quad\ldots\ldots\ldots&=0 \end{aligned}\right\}\ldots(b),$$

$$\left.\begin{aligned} (a\,C^2+c\,A^2-2e\,AC)\,x^2+2\,(eAB-dAC-eBC+fC^2)\,xy&\\ +\,(bC^2+cB^2-2dBC)\,y^2&=0\\ (a'C^2+c'A^2-2e'AC)\,x^2+\quad\ldots\ldots\ldots&=0 \end{aligned}\right\}\ldots(c).$$

We have to determine two conditions, which however cannot be perfectly expressed by two equations of conditions. From the pairs of equations (a), (b), (c), the conditions are obtained in various forms.

Thus, equating coefficients of (a), we have an unsymmetrical form (which fails if $A=0$); namely,

$$\frac{aBC-eAB-fAC+dA^2}{aB^2+bA^2-2fAB}=\frac{a'BC-e'AB-f'AC+d'A^2}{a'B^2+b'A^2-2f'AB},$$
$$\frac{aC^2+cA^2-2eAC}{aB^2+bA^2-2fAB}=\frac{a'C^2+c'A^2-2e'AC}{a'B^2+b'A^2-2f'AB}.$$

Identifying the ratios of the coefficients of y^2, z^2 in (a), and x^2, z^2 in (b), we get the system

$$\frac{aB^2+bA^2-2fAB}{a'B^2+b'A^2-2f'AB}=\frac{aC^2+cA^2-2eAC}{a'C^2+c'A^2-2e'AC}=\frac{bC^2+cB^2-2dBC}{b'C^2+c'B^2-2d'BC}.$$

And we similarly get the symmetrical system

$$\frac{a\mathrm{BC}-e\mathrm{AB}-f\mathrm{AC}+d\mathrm{A}^2}{a'\mathrm{BC}-e'\mathrm{AB}-f'\mathrm{AC}+d'\mathrm{A}^2}=\frac{b\mathrm{AC}-d\mathrm{AB}-f\mathrm{BC}+e\mathrm{B}^2}{b'\mathrm{AC}-d'\mathrm{AB}-f'\mathrm{BC}+e'\mathrm{B}^2}$$
$$=\frac{c\mathrm{AB}-d\mathrm{AC}-e\mathrm{BC}+f\mathrm{C}^2}{c'\mathrm{AB}-d'\mathrm{AC}-e'\mathrm{BC}+f'\mathrm{C}^2}.$$

These conditions in effect comprise (1) the conditions that a trilinear equation may represent a circle, (2) the conditions that a section of a quadric surface by a plane may be a circle.

And therefore they also yield the conditions for an umbilicus on a quadric surface; for instance, by making $d'=e'=f'=0$, $a'=b'=c'=1$, we get the general forms. For the special forms, when A or B or C $=0$, we must have recourse to the unsymmetrical systems. Thus the one written gives for $\mathrm{B}=0$

$$\mathrm{B}=0,\quad d\mathrm{A}-f\mathrm{C}=0,\quad a\mathrm{C}^2+c\mathrm{A}^2-2e\mathrm{AC}=b\,(\mathrm{A}^2+\mathrm{C}^2).$$

2652. (Proposed by J. J. Walker, M.A.)—If $l+m+n=0$, prove that $(l^2+m^2+n^2)^3-2\,(l-m)^2\,(m-n)^2\,(n-l)^2-54l^2m^2n^2$ vanishes; and hence verify the identity $\Delta=\mathrm{I}^3-27\mathrm{J}^2$, where Δ is the discriminant, and I, J are the fundamental invariants of $(a,b,c,d,e\between x,y)^4$.

Solution by the Proposer.

We have $l^2+m^2+n^2=-2\,(mn+nl+lm)$, since $l+m+n=0$;

therefore $(mn+nl+lm)^3=m^3n^3+n^3l^3+l^3m^3-3l^2m^2n^2$;

whence $(l^2+m^2+n^2)^3=-8\,(m^3n^3+n^3l^3+l^3m^3)+24l^2m^2n^2$ .... (1).

Again, $(m-n)^2\,(n-l)^2\,(l-m)^2$

$$\equiv\left\{(m+n)^2-4mn\right\}\left\{......\right\}\left\{......\right\}=(l^2-4mn)\,(m^2-4nl)\,(n^2-4lm)$$
$$\equiv-4\,(m^3n^3+n^3l^3+l^3m^3)+16lmn\,(l^3+m^3+n^3)-63l^2m^2n^2.$$

But since $l+m+n=0$, $l^3+m^3+n^3=3lmn$; therefore

$$(m-n)^2\,(n-l)^2\,(l-m)^2=-4\,(m^3n^3+n^3l^3+l^3m^3)-15l^2m^2n^2.$$

Eliminating $m^3n^3+n^3l^3+l^3m^3$ between this equation and (1), there results

$$(l^2+m^2+n^2)^3-2\,(l-m)^2\,(m-n)^2\,(n-l)^2-54l^2m^2n^2=0\ \text{.........}\ (2).$$

For the second part of the question, if $\alpha,\beta,\gamma,\delta$ are the roots of $(a,b,c,d,e\between x,y)^4$, and

$$l=(\alpha-\beta)\,(\gamma-\delta),\quad m=(\alpha-\gamma)\,(\delta-\beta),\quad n=(\alpha-\delta)\,(\beta-\gamma),$$

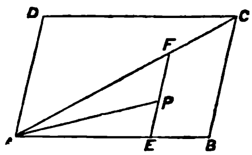
the relation $l+m+n=0$ will hold; but (Professor Cayley's Fifth *Memoir on Quantics*, Phil. Trans., Vol. 148)

$$\mathrm{I}=\frac{a^2}{24}\,(l^2+m^2+n^2),\quad \mathrm{J}=\frac{a^3}{432}\,(m-n)\,(n-l)\,(l-m),\quad\text{and}\quad \Delta=\frac{a^6}{256}\,l^2m^2n^2.$$

Hence, and from the relation (2), it is verified that $\mathrm{I}^3-27\mathrm{J}^2-\Delta=0$.

3005. (Proposed by S. Watson.)—Find the mean distance of all the points in a parallelogram from one of its angles.

I. *Solution by the* Proposer.



Let ABCD be the parallelogram, AC its diagonal, P any point in the triangle ABC, and EF a parallel to BC through P. Put $AB = a$, $BC = b$, $AC = d$, $AE = x$, $EP = y$; and $\angle DAB = \alpha$. Then $EF = \frac{bx}{a}$; an element of the triangle at $P = \sin\alpha \, dx\, dy$; hence, when P takes every position in the triangle ABC, the sum of the lengths AP is

$$\sin\alpha \int_0^a dx \int_0^{\frac{bx}{a}} (x^2 + y^2 + 2xy\cos\alpha)^{\frac{1}{2}} dy$$

$$= \tfrac{1}{6} a \sin\alpha \left\{ (b + a\cos\alpha)\, d - a^2 \cos\alpha + a^2 \sin^2\alpha \log \frac{b + a\cos\alpha + d}{a(1+\cos\alpha)} \right\} \ \dots (1).$$

The sum of the distances of all points in the triangle ADC from A is

$$\tfrac{1}{6} b \sin\alpha \left\{ (a + b\cos\alpha)\, d - b^2 \cos\alpha + b^2 \sin^2\alpha \log \frac{a + b\cos\alpha + d}{b(1+\cos\alpha)} \right\} \ \dots\dots (2).$$

Hence, the mean distance required is

$$\frac{(1)+(2)}{ab\sin\alpha} = \frac{1}{6}\left\{ 2d + \left(\frac{a^2+b^2}{ab}\right) d\cos\alpha - \left(\frac{a^3+b^3}{ab}\right)\cos\alpha \right.$$
$$\left. + \frac{a^2\sin^2\alpha}{b} \log \frac{b + a\cos\alpha + d}{a(1+\cos\alpha)} + \frac{b^2\sin^2\alpha}{a} \log \frac{a + b\cos\alpha + d}{b(1+\cos\alpha)} \right\} \ \dots\dots (4).$$

In the same way it may be shown, if $BD = d_1$, that the mean distance of all points in the parallelogram from B is

$$\frac{1}{6}\left\{ 2d_1 - \left(\frac{a^2+b^2}{ab}\right) d_1\cos\alpha + \left(\frac{a^3+b^3}{ab}\right)\cos\alpha \right.$$
$$\left. + \frac{a^2\sin^2\alpha}{b} \log \frac{b - a\cos\alpha + d_1}{a(1-\cos\alpha)} + \frac{b^2\sin^2\alpha}{a} \log \frac{a - b\cos\alpha + d_1}{b(1-\cos\alpha)} \right\} \ \dots\dots (5).$$

Hence, the mean distance of all the angles from every point in the parallelogram is

$$\frac{1}{12}\left\{ 2(d+d_1) + \left(\frac{a^2+b^2}{ab}\right)(d-d_1)\cos\alpha \right.$$
$$+ \frac{a^2\sin^2\alpha}{b} \log \frac{(b + a\cos\alpha + d)(b - a\cos\alpha + d_1)}{a^2\sin^2\alpha}$$
$$\left. + \frac{b^2\sin^2\alpha}{a} \log \frac{(a + b\cos\alpha + d)(a - b\cos\alpha + d_1)}{b^2\sin^2\alpha} \right\}.$$

II. *Solution by* Artemas Martin.

Let the polar coordinates of any point P on ABCD be $r = AP$, $\phi = \angle PAE$; M the average distance required; and M_1, M_2 the parts of it belonging to the triangles ABC, ADC respectively.

Then $M_1 = \dfrac{\iint r^2 dr d\phi}{2\iint r dr d\phi} = \dfrac{1}{ab\sin\beta}\iint r^2 dr d\phi = \dfrac{1}{3ab\sin\beta}\int r^3 d\phi.$

The limits of r are 0 and $\dfrac{a\sin\alpha}{\sin(\alpha-\phi)}$, and the limits of ϕ are 0 and

$$\sin^{-1}\left(\frac{b\sin\alpha}{d}\right) = \text{CAD} = \phi', \text{ suppose};$$

therefore
$$M_1 = \frac{a^2\sin^2\alpha}{3b}\int_0^{\phi'}\frac{d\phi}{\sin^3(\alpha-\phi)}.$$

Now $\displaystyle\int\frac{d\phi}{\sin^3(\alpha-\phi)} = -\int\frac{d\theta}{\sin^3\theta} = \tfrac{1}{2}\operatorname{cosec}\theta\cot\theta + \tfrac{1}{2}\log(\operatorname{cosec}\theta + \cot\theta)$,

by putting $\theta = \alpha - \phi = \text{PAD}$. Hence we have, finally,

$$M_1 = -\frac{a^2\cos\alpha}{6b} + \frac{d(d^2-a^2\sin^2\alpha)^{\frac{1}{2}}}{6b} + \frac{a^2\sin^2\alpha}{6b}\log\left(\frac{d+(d^2-a^2\sin^2\alpha)^{\frac{1}{2}}}{a(1+\cos\alpha)}\right).$$

Similarly

$$M_2 = -\frac{b^2\cos\alpha}{6a} + \frac{d(c^2-b^2\sin^2\alpha)^{\frac{1}{2}}}{6a} + \frac{b^2\sin^2\alpha}{6a}\log\left(\frac{d+(d^2-b^2\sin^2\alpha)^{\frac{1}{2}}}{b(1+\cos\alpha)}\right).$$

Therefore

$$M = M_1 + M_2 = -\frac{(a^3+b^3)\cos\alpha}{6ab} + \frac{ad(d^2-a^2\sin^2\alpha)^{\frac{1}{2}} + bd(d^2-b^2\sin^2\alpha)^{\frac{1}{2}}}{6ab}$$
$$+ \frac{a^2\sin^2\alpha}{6b}\log\left(\frac{d+(d^2-a^2\sin^2\alpha)^{\frac{1}{2}}}{a(1+\cos\alpha)}\right) + \frac{b^2\sin^2\alpha}{6a}\log\left(\frac{d+(d^2-b^2\sin^2\alpha)^{\frac{1}{2}}}{b(1+\cos\alpha)}\right).$$

[If we put $b + a\cos\alpha$ for $(d^2-a^2\sin^2\alpha)^{\frac{1}{2}}$, and $a + b\cos\alpha$ for $(d^2-b^2\sin^2\alpha)^{\frac{1}{2}}$, the above expression for M reduces to the one found in Mr. WATSON'S Solution.]

3183. (Proposed by A. MARTIN.)—Find three integral numbers in geometrical progression, such that if each be increased by unity the sums will be squares.

I. *Solution by* JUDGE SCOTT.

Let (x^2-1), $2x(x^2-1)$, $4x^2(x^2-1)$ be the three numbers in geometrical progression; then, since the first and third, when increased by unity, are evidently both squares, it only remains to make

$$2x^3 - 2x + 1 = \square = p^2, \quad \text{or} \quad 2x^3 - 2x = p^2 - 1.$$

This equation may be written $(2x+2)(x^2-x) = (p+1)(p-1)$.

Taking $2x+2 = p \pm 1$, and $x^2 - x = p \mp 1$, we have by subtraction

$$(2x+2) - (x^2-x) = \pm 2, \quad \text{whence} \quad x = 3 \text{ or } 4.$$

Taking $x = 3$ and 4, the numbers are 8, 48, 288; 15, 120, 960.

II. *Solution by the* PROPOSER.

Let x, xy, xy^2 be the numbers; then

$$x+1=\square,\quad xy+1=\square,\quad xy^2+1=\square\ldots\ldots\ldots\ldots(1,2,3).$$

Put $y=a^2x+2a$, then (2) is satisfied; and (3) becomes

$$a^4x^3+4a^3x^2+4a^2x+1=\square=(1+2a^2x)^2,\text{ suppose};$$

then $$x=\frac{4a-4}{a},\text{ and } x+1=\frac{5a-4}{a}=\square=b^2,\text{ suppose};$$

therefore $$a=\frac{4}{5-b^2}.$$

Take $b=2$, then $a=4$, $x=3$, $y=56$; and the numbers are 3, 168, 9408.
Take $b=3$, then $a=-1$, $x=8$, $y=6$; and the numbers are 8, 48, 288.

III. *Solution by* DR. DAVID S. HART.

In (1, 2, 3) above, let $x=m^2+2m$, then $x+1=(m+1)^2=\square$,

$$xy+1=(m^2+2m)y+1=\square,\quad xy^2+1=(m^2+2m)y^2+1=\square\ldots\ldots(4,5).$$

Let $y=\dfrac{n^2-2n}{m^2+2m}$, then (4) is $(n-1)^2=\square$; and (5) becomes

$$(m^2+2m)(n^2-2n)^2+(m^2+2m)^2=\square,$$

or $$(m^2+2m)n^4-4(m^2+2m)n^3+4(m^2+2m)n^2+(m^2+2m)^2=\square.$$

Put this $$=\left\{(m^2+2m)+2n^2\right\}^2,\text{ then } n=\frac{4(m^2+2m)}{m^2+2m-4}.$$

Take $m=1$, then $n=-12$, $y=56$, $x=3$; and the numbers are 3, 168, 9408.
Take $m=2$, then $n=8$, $y=6$, $x=8$; and the numbers are 8, 48, 288.

2852. (Proposed by J. J. WALKER, M.A.)—1. The locus of the centre of circles passing through four points, related as A, B, C, D in Questions 2803, 2833 (*Reprint*, Vol. XII., pp. 37, 90), is the conic polar reciprocal to the given conic with respect to the concentric circle bisecting the intervals between the centre and foci.

2. The normal at B and the line joining the centre of the circle which osculates the conic at that point with the centre of the circle ABCD make equal angles with the axis of the conic.

Solution by the PROPOSER.

1. If (x', y') be A, the circle ABCD has for equation

$$2(x^2+y^2)-(a^2-b^2)\left(\frac{x'x}{a^2}-\frac{y'y}{b^2}\right)-(a^2+b^2)=0.$$

Hence, if (ξ, η) be the centre of this circle,

$$4a\xi = (a^2-b^2)\frac{x'}{a}, \quad 4b\eta = (b^2-a^2)\frac{y'}{b}.$$

Squaring, $$16(a^2\xi^2+b^2\eta^2) = (a^2-b^2)^2,$$

a conic polar-reciprocal to the given with respect to the concentric circle the square of whose diameter is a^2-b^2.

2. If (x_1, y_1) be B, and (α, β) the centre of circle osculating the conic at B, it may be shown that

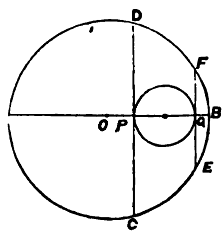
$$\alpha-\xi = \frac{3}{4}\frac{a^2-b^2}{a^2}x_1, \quad \beta-\eta = \frac{3}{4}\frac{b^2-a^2}{b^2}y_1,$$

whence $\dfrac{\beta-\eta}{\alpha-\xi} = -\dfrac{a^2y_1}{b^2x_1} = -$ tangent of angle which normal at (x_1, y_1) makes with axis.

It may be remarked that ξ, η and x', y' are *corresponding* points $\left(\text{since } \xi : x' = \dfrac{a^2-b^2}{4a} : a, \quad \eta : y' = \dfrac{a^2-b^2}{4b} : b\right)$; and that the line joining the centres of conic and circle ABCD is equally inclined to the axis as the normal at A, since $\eta : \xi = -a^2y' : b^2x'$.

3258. (Proposed by A. MARTIN.)—Two points are taken at random in the same radius of a given circle, and a third point anywhere in the surface of the circle. Find the chance that the triangle formed by joining these points is acute.

Solution by STEPHEN WATSON.



Let P, Q be the two points in the radius OB, through which draw CD, EF perpendicular to OB. Then in order that the triangle PQR may be acute, R must lie between CD and EF, but without the circle on PQ as diameter. As the chance will obviously be independent of the length of OB, put OB = 1, OP = x, OQ = y; then the area within which R may lie, is

$$\left\{\cos^{-1}x - x(1-x^2)^{\frac{1}{2}}\right\} - \left\{\cos^{-1}y - y(1-y^2)^{\frac{1}{2}}\right\} - (y-x)^2\pi \;\;\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also the number of positions of P, Q, R is π; hence doubling, because P and Q may be interchanged, the required chance p is

$$p = \frac{2}{\pi}\int_0^1 dx \int_x^1 dy\,(1)$$

$$= \frac{2}{\pi}\int_0^1 dx \left\{\cos^{-1}x - x(1-x^2)^{\frac{1}{2}} - \tfrac{2}{3}(1-x^2)^{\frac{3}{2}} - \tfrac{1}{3}\pi(1-x)^3\right\} = \frac{4}{3\pi} - \frac{5}{12}.$$

NOTE ON THE DIFFUSION OF GASES. *By* G. O. HANLON.

It was discovered experimentally by the late Mr. GRAHAM that, if we connect horizontally two vessels containing gases of different densities, the velocities of diffusion are inversely as the square roots of the specific gravities.

Let us suppose the two vessels to be placed far out in space, away from the influence of any exterior force. Let A be the area of the orifice connecting the gases; v, v' their velocities; and s, s' their specific gravities. Now, since the forces are all internal, the gases must move with such velocities as will not alter the position of their common centre of gravity when they come to rest by striking against the opposite sides of the vessels they respectively enter. In other words, the momentum of each opposing particle of gas must be equal at each instant. But the volumes of the gases passing through the orifice in any time are proportional to Av and Av', and the momentum of each of these is respectively Asv^2 and $As'v'^2$. Hence we have

$$Asv^2 = As'v'^2 \quad \text{or} \quad v : v' :: \sqrt{s'} : \sqrt{s},$$

which proves the result analytically.

3197. (Proposed by W. K. CLIFFORD, M.A.)—If the epicycloid described by a point on the circumference of a circle rolling on an equal fixed circle be loaded with matter proportional to its curvature at every point, the centre of gravity of the whole will be at the centre of the fixed circle.

Solution by the Rev. R. TOWNSEND, M.A., F.R.S.

Taking the cusp and axis of the epicycloid as origin and axis of x respectively, and denoting by a the radius of the fixed circle, by dm and $d\theta + ds$ the mass and curvature of any element ds of the curve, and by ω the polar angle of the point of contact of the revolving and fixed circles corresponding to the position of ds; then, since by hypothesis $dm = kd\theta = 3kd\omega$, and since, by the generation of the curve, $x = 4a\cos^2\omega\cos 2\omega$, therefore, for the entire curve, $m = 3k\pi$, and, for the distance $\bar{x}$ of its centre of gravity from the origin,

$$3k\pi \,.\, \bar{x} = 24ka \int_0^{\frac{1}{2}\pi} \cos^2\omega \cos 2\omega \, d\omega = 3k\pi \,.\, a,$$

from which, since at once $\bar{x} = a$, therefore, &c.

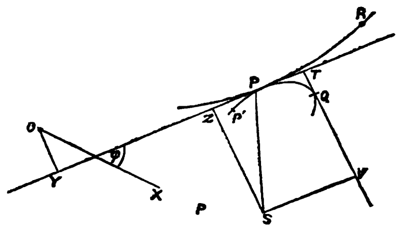
2980. (Proposed by Professor CROFTON, F.R.S.)—A given curve moves without rotation so as always to touch a fixed curve; in any position draw the common tangent, also a second tangent to the moving curve at right

angles to the former: find the envelope of the second tangent. Show that, if the intrinsic equation of the fixed curve be $\rho = f(\theta)$, and that of the moving curve $\rho = \phi(\theta)$, that of the required envelope will be

$$\rho = \frac{d}{d\theta} f(\theta - \tfrac{1}{2}\pi) - \frac{d}{d\theta}\phi(\theta - \tfrac{1}{2}\pi) + \phi(\theta);$$

where ρ in all cases stands for the radius of curvature, and θ for the inclination of the tangent to a fixed axis.

Solution by F. D. Thomson, M.A.



1. Suppose that, as in the figure, the convexities of the figure are opposed. Let S be a point fixed relatively to the moving curve PQ; O a fixed point; OX the initial line; TQV the tangent perpendicular to the common tangent PY. Draw the perpendiculars OY, SZ, SV as in the figure. Let $\mathrm{YT} = p$, $\mathrm{SZ} = p'$, $\mathrm{SV} = p''$, $\mathrm{OY} = \varpi$. Then $p = \mathrm{PY} + \mathrm{SV} - \mathrm{PZ}$. But if ϕ be the angle which PZ makes with the initial line, $\mathrm{PY} = \dfrac{d\varpi}{d\phi}$. Similarly $\mathrm{PZ} = -\dfrac{dp'}{d\phi}$, since p' decreases as ϕ increases; therefore $\quad p = \dfrac{d\varpi}{d\phi} + p'' + \dfrac{dp'}{d\phi};$

therefore $\quad \left(\dfrac{d^2p}{d\phi^2} + p = \dfrac{d^2p''}{d\phi^2} + p'' + \dfrac{d}{d\phi}\right)\dfrac{d\varpi^2}{d\phi^2} + \varpi + \dfrac{d^2p'}{d\phi^2} + p';$

therefore rad. curv. of envelope

$=$ rad. curv. at Q $+ \dfrac{d}{d\phi}$ (rad. curv. at P of the fixed curve

+ rad. curv. at P of the moving curve);

or, if $\rho = f(\phi)$ be the equation to PR, $\rho = \mathrm{F}(\phi)$ the equation to PQ,

$$\rho = \mathrm{F}(\phi + \tfrac{1}{2}\pi) + \frac{d}{d\phi}\left\{f(\phi) + \mathrm{F}(\phi)\right\};$$

or, writing ϕ for $\phi + \frac{1}{2}\pi$, the intrinsic equation to the envelope is

$$\rho = \mathrm{F}(\phi) + \frac{d}{d\phi}\left\{f(\phi - \tfrac{1}{2}\pi) + \mathrm{F}(\phi - \tfrac{1}{2}\pi)\right\}.$$

2. It may be easily shown that, with the same figure, the *locus* of S has for its equation $\qquad \rho = f(\phi) + \mathrm{F}(\phi);$

for if P′ be the point at which the tangent is parallel to the consecutive position of the common tangent at R, and S′ be the consecutive position of S, then SS′ is ultimately equal and parallel to RPP′. This gives $\mathrm{R} = \rho + \rho'$, where R is the rad. curv. at S, ρ at P of PR, ρ' at P of PQ. Hence the equation of the locus of S is $\rho = f(\phi) + \mathrm{F}(\phi)$.

3004. (Proposed by R. W. GENESE.)—A particle describes an ellipse about a centre of force in the focus S. Show that its mean distance from S is $a(1+\frac{1}{2}e^2)$.

Solution by the PROPOSER.

The mean distance $=\dfrac{\int \mathrm{SP}\,dt}{\int dt}=\dfrac{\int r^3 d\theta}{\int r^2 d\theta}$ (for $r\,d\theta = h\,dt$).

Now $$\int r^3 d\theta = \int \frac{l^3 d\theta}{(1-e\cos\theta)^3} = \int \frac{l^3 \sec^4 \frac{1}{2}\theta\, d\,(2\tan\frac{1}{2}\theta)}{\left\{1-e+(1+e)\tan^2\frac{1}{2}\theta\right\}^3},$$

which, from $\theta=0$ to $\theta=\pi$, $$= \int_0^\infty \frac{2l^3(1+x^2)^2\,dx}{\left\{1-e+(1+e)x^2\right\}^3},$$

$$= \int_0^\infty \frac{2l^3}{(1-e^2)^{\frac{5}{2}}}\,\frac{\left\{1+e+(1-e)y^2\right\}^2}{(1+y^2)^3}\,dy \quad \text{[by putting } (1+e)x^2=(1-e)y^2]$$

$$= \frac{2l^3}{(1-e^2)^{\frac{5}{2}}}\int_0^{\frac{1}{2}\pi}\left\{(1+e)\cos^2\theta+(1-e)\sin^2\theta\right\}^2 d\theta$$

$$= \frac{2l^3}{(1-e^2)^{\frac{5}{2}}}\left\{2(1+e^2)\,\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}+(1-e^2)\,\frac{1}{4}\cdot\frac{\pi}{2}\right\} = \frac{\pi l^3}{(1-e^2)^{\frac{5}{2}}}\left(1+\frac{e^2}{2}\right).$$

Again, $\int_0^\pi r^2 d\theta =$ area of ellipse $=\pi a^2(1-e^2)^{\frac{1}{2}}$, and $l=a(1-e^2)$;

therefore the mean distance is $a(1+\frac{1}{2}e^2)$.

2956. (Proposed by W. ROBERTS, Jun.)—Find the locus of a point such that, if from it tangents be drawn to a central conic, the product of these tangents shall be in a constant ratio to the product of the focal distances.

Solution by R. TUCKER, M.A.

Referring to the figure to the Solution of Quest. 2949, we have, since

$$\text{perpendicular from P on QR} = \frac{\mu-a^2b^2}{(a^4y'^2+b^4x'^2)^{\frac{1}{2}}},$$

$$\triangle\,\mathrm{PQR} = \tfrac{1}{2}\delta\,.\,\frac{\mu-a^2b^2}{(a^4y'^2+b^4x'^2)^{\frac{1}{2}}} = \frac{(\mu-a^2b^2)^{\frac{3}{2}}}{\mu};$$

also $= \frac{1}{2}PQ \, . \, PR \sin\theta = \dfrac{(\mu - a^2b^2)^{\frac{1}{2}}}{SP \, . \, HP} PQ \, . \, PR$; (Salmon's *Conics*, p. 198).

$$= \lambda (\mu - a^2b^2)^{\frac{1}{2}}, \text{ if } PQ \, . \, PR = \lambda \, . \, SP \, . \, HP;$$

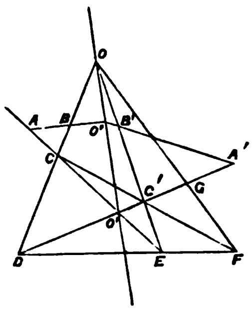
hence $$\mu - a^2b^2 = \lambda\mu, \quad \text{or} \quad \mu(1-\lambda) = a^2b^2,$$

where $$\mu = a^2y'^2 + b^2x'^2,$$

the equation to an ellipse, whose axes are $\dfrac{a}{(1-\lambda)^{\frac{1}{2}}}, \; \dfrac{b}{(1-\lambda)^{\frac{1}{2}}}$.

2957. (Proposed by G. A. Ogilvie.)—ABC and A′B′C are two copolar triangles. Let the straight line formed by joining the points of intersection of CB and C′A′, and of C′B′ and CA be denoted by DE. Let O be the intersection of BC and B′C′, then the straight line joining O to the intersection of CC′ and DE is the fourth harmonic to BC, B′C′ and the axis of the triangles.

Solution by James Dale.



Let ABC, A′B′C′ be the triangles, OO′O″ the axis, completing the figure as directed; let DE and CC′ intersect in F, and C′A′ and OF in G.

Then the diagonal DC′ of the complete quadrilateral OCDEC′F is harmonically divided by the other two diagonals in O′, G; therefore the pencil OD, OE, OO′, OF is harmonic.

2151. (Proposed by S. Roberts, M.A.)—In a Cartesian oval with a finite node (a nodal Limaçon), the difference of the lengths of the loops is four times the distance between the vertices.

Solution by the Rev. J. Wolstenholme, M.A.

If $r = a(e + \cos\theta)$ be the equation of the curve, the lengths of the loops

are $\quad 4a(1+e)\displaystyle\int_0^{\pi}(1 - c^2\sin^2\theta)^{\frac{1}{2}}\,d\theta, \quad$ and $\quad 4a(1+e)\displaystyle\int_0^{\pi}(1 - c^2\sin^2\theta)^{\frac{1}{2}}\,d\theta,$

where $\qquad c^2 = \dfrac{4e}{(1+e)^2}$, and $\tan^2 \alpha = \dfrac{1+e}{1-e}$;

and therefore the difference of their lengths is

$$2a(1+e)\left\{2\mathrm{E}(c, \alpha) - \mathrm{E}(c, \tfrac{1}{2}\pi)\right\}.$$

But, if $\qquad \cos\theta\cos\phi - \sin\theta\sin\phi\,(1-c^2\sin^2\mu)^{\frac{1}{2}} = \cos\mu,$

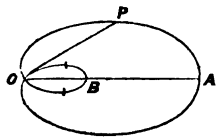
$$\mathrm{E}(c, \theta) + \mathrm{E}(c, \phi) - \mathrm{E}(c, \mu) = c^2 \sin\theta \sin\phi \sin\mu;$$

and the former relation is satisfied by $\theta = \phi = \alpha$, $\mu = \frac{1}{2}\pi$; therefore

$$2\mathrm{E}(c, \alpha) - \mathrm{E}(c, \tfrac{1}{2}\pi) = c^2 \sin^2\alpha = \frac{4e}{(1+e)^2}\cdot\frac{1+e}{2} = \frac{2e}{1+e},$$

or the difference of the lengths of the loops is $8ae$, which is four times the distance between the vertices. Applying the same to the ellipse, we get that if B, A be the ends of the axes, and L a point whose eccentric angle is $\tan^{-1}\left(\dfrac{b}{a}\right)^{\frac{3}{2}}$, $\qquad$ BL − AL = AC − BC.

In general, we shall have the same relation, if O be the node, A, B the vertices, P, Q points on the two loops such that

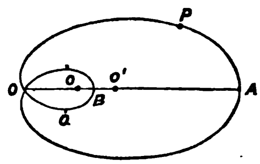


$$\frac{1}{\mathrm{OP}} - \frac{1}{\mathrm{OA}} = \frac{1}{\mathrm{OQ}} - \frac{1}{\mathrm{OB}},$$

$$\text{arc AP} - \text{arc BQ} = 2\mathrm{AB}\left(1 - \frac{\mathrm{OP}\,.\,\mathrm{OQ}}{\mathrm{OA}\,.\,\mathrm{OB}}\right)^{\frac{1}{2}},$$

which follows in exactly the same way from the general property of elliptic functions of the second order.

If P, Q be two such points (on opposite sides of the axis), the straight line bisecting the external angle between OP and OQ will meet PQ in a point lying on the circle



$$r = \frac{a(1-e^2)}{e^2}\cos\theta.$$

If O′ be taken on the axis such that $\mathrm{OO}' = \frac{1}{2}a(1-e^2)$, O′ is another form of the Limaçon, and the relation will be found to be equivalent to O′P . O′Q = O′A . O′B.

3213. (Proposed by the Rev. A. F. Torry, M.A.)—If three circles intersect two and two in the points A, A′; B, B′; C, C′; and through any other point O circles OAA′, OBB′, OCC′ be described, these will all meet again in another point.

I. *Solution by* Archer Stanley.

The point O being taken as origin, it is readily seen that the theorem is simply the inverse of the well-known one that the three radical axes of three given circles meet in a point; viz., their radical centre.

II. *Solution by the* REV. J. WOLSTENHOLME, M.A.

Let K be the radical centre of the three circles; and let O, O′ be the points of intersection of two circles through A, A′, and B, B′ respectively. Then AA′, BB′, OO′ meet in a point, and therefore in K; and KO . KO′ = KA . KA′ = KC . KC′; or, O, O′, C, C′ lie on one circle.

2360. (Proposed by the EDITOR.)—Find the average area of all the triangles in which there are given—(1) an angle and the sum of the two containing sides, all possible values of these sides being considered equally probable; (2) the sum of two sides only, all possible values of these two sides, and of the included angle, being considered equally probable; (3) one side and the sum of the other two; (4) the sum of the three sides; all possible values of the sides in (3) and (4) being considered equally probable. Also show (5) that the average of the averages in (3) is seven-eighths of the average in (4).

Solution by T. SAVAGE, M.A.; *the* REV. J. WOLSTENHOLME, M.A.; S. WATSON, M.A.; *and others.*

1. Let l be the sum of the two sides, θ the contained angle. The average area is evidently

$$\int_0^l \tfrac{1}{2}x(l-x)\sin\theta\,dx \div \int_0^l dx = \frac{l^2\sin\theta}{12}.$$

2. If the value of θ be not given, the average is

$$\frac{l^2}{12}\int_0^\pi \sin\theta\,d\theta \div \int_0^\pi d\theta = \frac{l^2}{6\pi}.$$

3. Let the third side be given and equal to a. Then the area of the triangle is $\tfrac{1}{4}(l^2-a^2)^{\frac{1}{2}}\left\{a^2-(l-2x)^2\right\}^{\frac{1}{2}}$, where x is one of the other sides. Now $a+x > l-x$ and $a+l-x > x$, which give $\tfrac{1}{2}(l-a)$ and $\tfrac{1}{2}(l+a)$ as the limits of x [*i.e.*, the measure of the number of such triangles is a]. Hence the average area of the triangle is

$$\frac{\tfrac{1}{4}(l^2-a^2)^{\frac{1}{2}}}{a}\int_{\frac{1}{2}(l-a)}^{\frac{1}{2}(l+a)}\left\{a^2-(l-2x)^2\right\}^{\frac{1}{2}}dx = \tfrac{1}{16}\pi a\,(l^2-a^2)^{\frac{1}{2}}.$$

4. Let half the sum of the sides be given ($=s$), and let x, y be two of the sides. Then x and y are both less than s, and $x+y > s$; thus, the limits of y being $(s-x)$ and s, while those of x are 0 and s, the average area is

$$\iint\left\{s(s-x)(s-y)(x+y-s)\right\}^{\frac{1}{2}}dx\,dy \div \iint dx\,dy$$

$$= \frac{2}{s^2}\int_0^s\left\{s(s-x)\right\}^{\frac{1}{2}}\left\{\int_0^x\left\{x(s-y)-(s-y)^2\right\}^{\frac{1}{2}}d(s-y)\right\}dx$$

$$= \frac{2}{s^2}\int_0^s\left\{s(s-x)\right\}^{\frac{1}{2}}\tfrac{1}{8}\pi x^2\,dx = \tfrac{4}{105}\pi s^2.$$

[The average of the averages in case (1) gives, of course, the average required in case (2); since all the averages in (1) are the representatives of the same number of triangles, and therefore enter with equal weights into the average for (2).

5. The average of the averages in case (3) is obtained by putting, in Mr. SAVAGE's expression for (3), $2s-a$ for l, integrating between the limits 0 and s of a, and dividing by s, the number of such averages. The result is

$$\frac{\pi}{8s^{\frac{1}{2}}}\int_0^s (s-a)^{\frac{1}{2}}\, a\,da = \frac{\pi}{8s^{\frac{1}{2}}}\int_0^s a^{\frac{1}{2}}\,(s-a)\,da = \tfrac{1}{30}\pi s^2,$$

which is *seven-eighths* of the true average in (4).

In (3) the averages do not enter with equal weights into the result. If we multiply the average in (3) by its weight, viz. a, which is the measure of the number of triangles of which this average is the representative, we find in this way, for the average required in (4),

$$\frac{\pi s^{\frac{1}{2}}}{8}\int_0^s (s-a)^{\frac{1}{2}}\, a^2 da \div \int_0^s a\,da = \frac{\pi}{4s^{\frac{3}{2}}}\int_0^s a^{\frac{1}{2}}\,(s-a)^2\,da = \tfrac{4}{105}\pi s^2,$$

which agrees with the result otherwise obtained by Mr. SAVAGE and others.]

3249. (Proposed by Prof. CAYLEY.)—Given on a given conic two quadrangles PQRS and $pqrs$, having the same centres, and such that P, p; Q, q; R, r; S, s are the corresponding vertices (that is, the four lines PQ, RS, pq, rs, all pass through the same point; and similarly the lines PR, QS, pr, qs, and the lines PS, QR, ps, qr): it is required to show that a conic may be drawn through the points p, q, r, s, touched at these points by the lines pP, qQ, rR, sS, respectively.

I. *Solution by* DR. HIRST, F.R.S.

Since PQ, RS, pq, rs are four concurrent chords of a conic, their extremities, by a well-known theorem, are conjugate points of an involution of points on the conic. Hence, if we indicate by (Pqrs) the anharmonic ratio of a pencil whose centre is *anywhere* on the conic, and whose rays pass through P, q, r, s, we shall have

$$(\mathrm{P}qrs) = (\mathrm{Q}psr).$$

In like manner, since PR, QS, pr, qs are concurrent,

$$(\mathrm{P}qrs) = (\mathrm{R}spq);$$

and since PS, QR, ps, qr are likewise concurrent,

$$(\mathrm{P}qrs) = (\mathrm{S}rqp).$$

Hence, taking p, q, r, s as centres of four pencils, we have

$$p\,(\mathrm{P}qrs) = q\,(\mathrm{Q}psr) = r\,(\mathrm{R}spq) = s\,(\mathrm{S}rqp) \quad\ldots\ldots\ldots\ldots\ldots (1).$$

Again, since there is but one conic which can be circumscribed to a given quadrangle $pqrs$ so as to touch a given line at one of the four vertices, and since the three tangents to this conic at the remaining three vertices are

perfectly determined, it is obvious that the tangents at the vertices to one and the same conic circumscribed to the quadrangle are corresponding rays of four homographic pencils; and conversely, corresponding rays of these four pencils touch at their centres one and the same circumscribed conic. The homography is determined by any three sets of four corresponding rays,—in other words, by means of any three circumscribed conics. Now if for this purpose we make use of the three line-pairs $(\overline{pq}, \overline{rs})$, $(\overline{pr}, \overline{sq})$, and $(\overline{ps}, \overline{qr})$, included in the pencil of circumscribed conics, it will be obvious that to the rays pq, pr, ps of the pencil $[p]$ will correspond respectively the rays qp, qs, qr of the pencil $[q]$, the rays rs, rp, rq of the pencil $[r]$, and the rays sr, sq, sp of the pencil s. But the relation (1) shows that of the four pencils whose homography is thus determined, pP, qQ, rR, sS are corresponding rays; hence they touch one and the same circumscribed conic at the respective centres p, q, r, s.

II. *Solution by the* PROPOSER.

Taken the centres for the vertices of the fundamental triangle, the equation of the given conic may be taken to be $x^2+y^2+z^2=0$; and then the coordinates of P, Q, R, S to be (A, B, C), (A, $-$B, C), (A, B, $-$C), (A, $-$B, $-$C) respectively, where $A^2+B^2+C^2=0$; and those of p, q, r, s to be (α, β, γ), $(\alpha, -\beta, \gamma)$, $(\alpha, \beta, -\gamma)$, $(\alpha, -\beta, -\gamma)$ respectively, where $\alpha^2+\beta^2+\gamma^2=0$. The required conic, assuming it to exist, will be given by an equation of the form $lx^2+my^2+nz^2=0$. This must pass through the point (α, β, γ), and the tangent at this point must be $x(B\gamma-C\beta)+y(C\alpha-A\gamma)+z(A\beta-B\alpha)$; that is, we must have $l\alpha^2+m\beta^2+n\gamma^2=0$, and $l\alpha : m\beta : n\gamma = B\gamma-C\beta : C\alpha-A\gamma : A\beta-B\alpha$. The first condition is obviously included in the second; and the second condition remains unaltered if we reverse the signs of B, β, or of C, γ, or of B, β and C, γ. Hence the conic passing through p, and touched at this point by pP, will also pass through the points q, r, s, and be touched at these points by the lines qQ, rR, sS, respectively; that is, the equation of the required conic is

$$\frac{B\gamma-C\beta}{\alpha}x^2+\frac{C\alpha-A\gamma}{\beta}y^2+\frac{A\beta-B\alpha}{\gamma}z^2=0;$$

or, what is the same thing,

$$\begin{vmatrix} \beta\gamma x^2, & \gamma\alpha y^2, & \alpha\beta z^2 \\ A\ , & B\ , & C \\ \alpha\ , & \beta\ , & \gamma \end{vmatrix} = 0.$$

3049. (Proposed by R. W. GENESE.)—A straight line OX touches a conic at O, and from four points P, Q, R, S on the line tangents are drawn to the conic, making angles α, β, γ, δ respectively with OX; show that with the usual convention as to sign

$$\frac{\mathrm{OP}\cot\alpha}{\mathrm{PQ}.\mathrm{PR}.\mathrm{PS}}+\frac{\mathrm{OQ}\cot\beta}{\mathrm{QP}.\mathrm{QR}.\mathrm{QS}}+\frac{\mathrm{OR}\cot\gamma}{\mathrm{RP}.\mathrm{RQ}.\mathrm{RS}}+\frac{\mathrm{OS}\cot\delta}{\mathrm{SP}.\mathrm{SQ}.\mathrm{SR}}=0.$$

Solution by the PROPOSER.

Let TV be the tangent from any point T on OX,

$$OT = x, \quad \text{the angle VTX} = \omega;$$

then there must exist some relation of the form

$$\tan\omega = \frac{ax^2+bx+c}{a'x^2+b'x+c'} \quad \text{.............................(1)},$$

since for every value of x there is one only of $\tan\omega$, whilst for every value of $\tan\omega$ (from 0 through ∞ to 0 again) there are two of x (for parallel tangents).

Now, when $x=0$, $\omega=0$, therefore $c=0$;

and when $x=\infty$, $\omega=\pi$, therefore $a=0$;

thus (1) reduces to the form (easily verified by x, y equations)

$$a'x^2+b'x+c = bx\cot\omega.$$

Taking four values of x corresponding to P, Q, R, S, and eliminating a', b', c', b, we get

$$\begin{vmatrix} x_1^2, & x_1, & 1, & x_1\cot\alpha \\ x_2^2, & x_2, & 1, & x_2\cot\beta \\ x_3^2, & x_3, & 1, & x_3\cot\gamma \\ x_4^2, & x_4, & 1, & x_4\cot\delta \end{vmatrix} = 0,$$

which, since $\begin{vmatrix} x_2^2, & x_2, & 1 \\ x_3^2, & x_3, & 1 \\ x_4^2, & x_4, & 1 \end{vmatrix} \equiv (x_2-x_3)(x_3-x_4)(x_2-x_4)$, &c.,

is seen to be the relation in question.

NOTE.—If the curve be a parabola, we may take one of the tangents at an infinite distance, and the relation reduces to

$$\frac{OP\cot\alpha}{PQ\,.\,PR} + \frac{OQ\cot\beta}{QP\,.\,QR} + \frac{OR\cot\gamma}{RP\,.\,RQ} = 0;$$

and this relation may be established independently, for in place of equation (1) we shall have $\tan\omega = \dfrac{bx+c}{b'x+c'}$, and the rest is obvious.

3169. (Proposed by the Rev. J. BLISSARD.)—1. Let U be a Representative Quantity which satisfies the equation

$$U(U-1)(U-2)\ldots(U-n+1) = 1;$$

then $$U^n\,(=U_n) = \left(\frac{\Delta}{1} + \frac{\Delta^2}{1\,.\,2} + \frac{\Delta^3}{1\,.\,2\,.\,3} + \ldots\ldots + \frac{\Delta^n}{1\,.\,2\ldots n}\right)0^n.$$

2. Let $U(U+1)\ldots(U+n-1) = 1$, then

$$U^n\,(=U_n) = (-1)^{n+1}\left(\frac{\Delta}{1} - \frac{\Delta^2}{1\,.\,2} + \frac{\Delta^3}{1\,.\,2\,.\,3} - \ldots\ldots \pm \frac{\Delta^n}{1\,.\,2\ldots n}\right)0^n.$$

Solution by the PROPOSER.

The above may be regarded as an exercise in Representative Notation.

1. Here $(1+x)^{U}-1$, which $= Ux + \frac{U(U-1)}{1.2}x^2 + \ldots\ldots$,

becomes $x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots\ldots$, that is, $= \epsilon^x - 1$.

For $1+x$ put ϵ^x, then we have

$$\epsilon^{Ux}-1 = \epsilon^{\epsilon^x-1}-1 = \frac{1}{\epsilon} - 1 + \frac{1}{\epsilon}\left(\epsilon^x + \frac{\epsilon^{2x}}{1.2} + \frac{\epsilon^{3x}}{1.2.3} + \ldots\ldots\right).$$

Hence, equating coefficients of x^n, we obtain

$$U^n (= U_n) = \frac{1}{\epsilon}\left(1^n + \frac{2^n}{1.2} + \frac{3^n}{1.2.3} + \ldots\ldots\right)$$

$$= \frac{\Delta^0}{\epsilon}\left\{\frac{1+\Delta}{1} + \frac{(1+\Delta)^2}{1.2} + \frac{(1+\Delta)^3}{1.2.3} + \ldots\ldots\right\} 0^n \quad [\text{since } m^n = \Delta^0(1+\Delta)^m 0^n]$$

$$= \frac{\Delta^0}{\epsilon}\left(\epsilon^{1+\Delta}-1\right) 0^n = \Delta^0\left(\epsilon^{\Delta} - \frac{1}{\epsilon}\right) 0^n$$

$$= \left(\frac{\Delta}{1} + \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} + \ldots\ldots + \frac{\Delta^n}{1.2\ldots n}\right) 0^n$$

$$[\text{since } \Delta^0 0^n = 0 \text{ and } \Delta^{m+n} 0^n = 0].$$

2. Since $U(U+1)\ldots(U+n-1) = 1$, then

$(1+x)^{-U}-1$, which $= -Ux + \frac{U(U+1)}{1.2}x^2 - \frac{U(U+1)(U+2)}{1.2.3}x^3 + \ldots$,

becomes $-x + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \ldots\ldots$, that is, $= \epsilon^{-x}-1$.

For $1+x$ put ϵ^x, then we have

$$\epsilon^{-Ux}-1 = \epsilon^{1-\epsilon^x}-1 = \epsilon - 1 - \epsilon\left(\epsilon^x - \frac{\epsilon^{2x}}{1.2} + \frac{\epsilon^{3x}}{1.2.3} - \ldots\ldots\right).$$

Hence, equating coefficients of x^n, we obtain

$$(-1)^{n+1}U^n = \epsilon\left(1^n - \frac{2^n}{1.2} + \frac{3^n}{1.2.3} - \ldots\ldots\right)$$

$$= \epsilon\left\{\frac{1+\Delta}{1} - \frac{(1+\Delta)^2}{1.2} + \frac{(1+\Delta)^3}{1.2.3} - \ldots\ldots\right\} 0^n;$$

$$\therefore\ U^n (= U_n) = (-1)^{n+1}\epsilon\Delta^0\left(1-\epsilon^{-(1+\Delta)}\right) 0^n = (-1)^{n+1}\Delta^0\left(\epsilon - \epsilon^{-\Delta}\right) 0^n$$

$$= (-1)^{n+1}\left(\frac{\Delta}{1} - \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} - \ldots\ldots \pm \frac{\Delta^n}{1.2\ldots n}\right) 0^n.$$

COR. 1.—In $U(U-1)\ldots(U-n+1)$, for n put successively $1, 2, 3, \ldots$; then $U_1 = 1$, $U_2 - U_1 = 1$, therefore $U_2 = 2$; $U_3 - 3U_2 + 2U_1 = 1$, therefore $U_3 = 5$; similarly $U_4 = 15$, &c. Therefore, from (1),

$$U_4 = \left(\Delta + \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} + \frac{\Delta^4}{1.2.3.4}\right) 0^4 = 1+7+6+1 = 15.$$

From $U(U+1)\dots(U+n-1)$ we obtain $U_1=1$, $U_2=0$, $U_3=-1$, $U_4=1$, $U_5=2$, &c.; therefore, from (2),

$$U_5 = \left(\Delta - \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} - \frac{\Delta^4}{1.2.3.4} + \frac{\Delta^5}{1.2.3.4.5}\right) 0^5$$
$$= 1-15+25-10+1 = 2.$$

COR. 2.—In $(1+x)^U = \epsilon$ [obtained from $U(U-1)\dots(U-n+1) = 1$] put $\frac{-x}{1+x}$ for x, then we obtain

$$(1+x)^{-U} = \epsilon^{\frac{-x}{1+x}} = 1 - \frac{x}{1+x} + \frac{1}{1.2}\left(\frac{x}{1+x}\right)^2 - \dots\dots;$$

and equating coefficients of x^n, we have

$$\frac{U(U+1)\dots(U+n-1)}{1.2\dots n}$$
$$= 1 + \frac{1}{2}\,\frac{n-1}{1^2} + \frac{1}{3}\,\frac{(n-1)(n-2)}{(1.2)^2} + \frac{1}{4}\,\frac{(n-1)(n-2)(n-3)}{(1.2.3)^2} + \dots\dots.$$

Similarly, from $U(U+1)\dots(U+n-1) = 1$, by a like process we obtain

$$\frac{U(U-1)\dots(U-n+1)}{1.2\dots n} = (-1)^{n+1}\left\{1 - \frac{1}{2}\,\frac{n-1}{1^2} + \frac{1}{3}\,\frac{(n-1)(n-2)}{(1.2)^2} - \dots\dots\right\}.$$

2279. (Proposed by G. O. HANLON.)—An ellipse is described having its major axis three times the minor. Any parabola is drawn satisfying the following conditions:—The intercept made on its principal axis by the axes of the ellipse is constant, and equal to the sum of the semi-axes of the ellipse; its focus lies on the ellipse, in the same quadrant as the intercept, and the tangent at its vertex passes through the foot of the perpendicular from the middle of the intercept on the major axis of the ellipse. An equilateral hyperbola is also drawn, whose axis is equal to the intercept, and whose axis and centre fall on the major axis and centre of the ellipse. Now if any tangent is drawn to the parabola, and from the points in which it meets the hyperbola lines are drawn parallel to the asymptotes, one of the loci of their intersections envelopes a circle.

Solution by the PROPOSER.

Let $2a$ be the sum of the semi-axes of the ellipse, and θ the angle between the minor axis and the intercept. Taking the centre for origin, it is easy to show that the distance from the focus to the vertex of the parabola $= \frac{1}{2}a \cos 2\theta$. The parameter therefore $= 2a \cos 2\theta = -2a\,\frac{m^2-1}{m^2+1}$, if $m = \tan\theta$. The equation to the principal axis is $x - my + 2a \sin\theta = 0$,

and that to the tangent at the vertex is $mx+y+ma\sin\theta = 0$. The equation to the parabola is therefore

$$\left\{\frac{x-my+2a\sin\theta}{(1+m^2)^{\frac{1}{2}}}\right\}^2 = -2a\,\frac{m^2-1}{m^2+1}\left\{\frac{mx+y+ma\sin\theta}{(1+m^2)^{\frac{1}{2}}}\right\}.$$

Substituting for $\sin\theta$ its value $\dfrac{m}{(1+m^2)^{\frac{1}{2}}}$, this equation becomes

$$\left\{x-my+ma\,(1+m^2)^{\frac{1}{2}}\right\}^2 = (1-m^2)\left\{2ay\,(1+m^2)^{\frac{1}{2}}-m^2a^2\right\}.$$

Any tangent to this is of the form

$$r^2(1-m^2)-2r\left\{x-my+ma\,(1+m^2)^{\frac{1}{2}}\right\}+2ay\,(1+m^2)^{\frac{1}{2}}-m^2a^2 = 0,$$

where r is an auxiliary symbol.

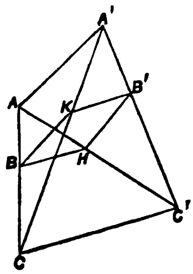
Let $mr+a\,(1+m^2)^{\frac{1}{2}} = s$, and the equation to the tangent reduces to $r^2-s^2-2xr+2ys+a^2 = 0$. The points where this tangent meets the hyperbola will be found to be

$$x' = \frac{(r+s)^2+a^2}{2\,(r+s)} \quad\text{and}\quad y' = \frac{(r+s)^2-a^2}{2\,(r+s)},$$

$$x'' = \frac{a^2+(r-s)^2}{2\,(r-s)} \quad\text{and}\quad y'' = \frac{a^2-(r-s)^2}{2\,(r-s)}.$$

The equations to two of the lines through these points parallel to the asymptotes are, after reducing, $y-x = s-r$ and $y+x = s+r$. The points of intersection are therefore $y=s$ and $x=r$; but $s = mr+a\,(1+m^2)^{\frac{1}{2}}$, the equation to the tangent to a circle central with the ellipse, and having its diameter equal to the intercept.

3268. (Proposed by R. W. Genese.)—ABC, A′B′C′ are two straight lines so divided that AB : BC = A′B′ : B′C′; show that parallels through B, B′ to AA′, CC′ meet on AC′, A′C.



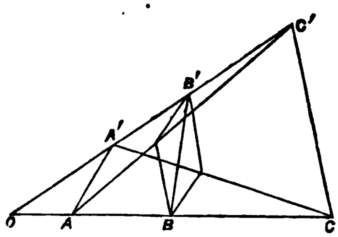
I. *Solution by the* Rev. G. H. Hopkins, M.A.

Let BH be drawn parallel to CC′ to meet AC′ in H; and join B′H. Then we have

$$AB : BC = AH : HC' = A'B' : B'C;$$

therefore BH is parallel to AA′. In a similar way, BK and B′K, drawn respectively parallel to AA′ and CC′, intersect on A′C.

II. *Solution by* STEPHEN WATSON.



Produce CA, C′A′ to meet in O; take OC, OC′ as axes; and put $OA = a$, $OA' = b$, $OB = x_1$, $OB' = y_1$, $OC = x_2$, $OC' = y_2$. Then $x_1 : x_2 = y_1 : y_2$, or $x_1 : x_1 + x_2 = y_1 : y_1 + y_2$.

The equations of the parallel to CC′ through B, and the line AC′, are respectively

$$(a + x_1 + x_2)\, y + (b + y_1 + y_2)\, x = (a + x_1)(b + y_1 + y_2) \;\ldots(1),$$

$$ay + (b + y_1 + y_2)x = a(b + y_1 + y_2) \ldots(2);$$

therefore $\qquad (x_1 + x_2)\, y = x_1 (b + y_1 + y_2) = bx_1 + y_1 (x_1 + x_2),$

and $\qquad (x_1 + x_2)\, x = ax_2; \qquad$ therefore $\;ay + bx = a(b + y_1),$

the equation of the parallel through B′ to AA′.

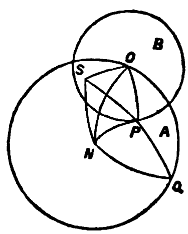
In like manner, parallels through B, B′ to AA′ and CC′ respectively, meet on A′C.

3133. (Proposed by SAMUEL ROBERTS, M.A.)—Given two small circles of a sphere, one of which (A) passes through the centre of the other (B). An arc of a great circle moves on the sphere with one extremity on (A) and its middle point on (B), and its length is equal to the diameter arc of (B). The locus of the other extremity is represented by the equation

$$(\cos^2 \tfrac{1}{2}\rho - \cos^2 r)(\cos a \cos \tfrac{1}{2}\rho - \sin a \sin \tfrac{1}{2}\rho \cos \theta)^2 - \sin^2 a \cos^2 r \sin^2 \theta = 0,$$

together with the small circle $\cos^2 \frac{1}{2}\rho - \cos^2 r = 0$ (the centre of B being the pole). If we only include quantities up to the second order dependent on the spherical form of the surface, what is the nature of the approximately plane curve traced?

I. *Solution by* J. J. WALKER, M.A.



Let N, O be the centres, and a, r the spherical radii of the circles (A), (B) respectively; P the middle point of the arc $QPS = 2r$; $OS = \rho$, and the angle $SON = \theta$. Then

$$\cos NS = \cos \rho \cos a + \sin \rho \sin a \cos \theta,$$

also $\qquad \cos NS = 2 \cos r \cos NP - \cos a,$

whence

$$\cos^2 \tfrac{1}{2}\rho \cos a + \cos \tfrac{1}{2}\rho \sin \tfrac{1}{2}\rho \sin a \cos \theta = \cos r \cos NP$$
$$= \cos r \left\{ \cos a \cos r + \sin a \sin r \cos (POS - \theta) \right\},$$

or

$$(\cos^2 \tfrac{1}{2}\rho - \cos^2 r) \cos a + \cos \tfrac{1}{2}\rho \sin \tfrac{1}{2}\rho \sin a \cos \theta$$
$$= \cos r \sin r \sin a\, (\cos POS \cos \theta + \sin POS \sin \theta);$$

but $\quad \cos \mathrm{POS} = \tan \frac{1}{2}\rho \cot r, \quad \sin \mathrm{POS} = \dfrac{(\cos^2 \frac{1}{2}\rho - \cos^2 r)^{\frac{1}{2}}}{\cos \frac{1}{2}\rho \sin r}.$

Substituting these values in the equation above, it becomes

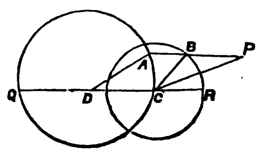
$$(\cos^2 \tfrac{1}{2}\rho - \cos^2 \rho)(\cos a \cos \tfrac{1}{2}\rho + \sin a \sin \tfrac{1}{2}\rho \cos\theta)$$
$$= (\cos^2 \tfrac{1}{2}\rho - \cos^2 r)^{\frac{1}{2}} \sin a \cos r \sin\theta.$$

Finally, squaring, and changing θ into its supplement, the locus of S, as given in the question, is obtained.

Writing for $\cos \frac{1}{2}\rho$, $1-\dfrac{\rho^2}{4a^2}$; for $\cos r$, $1-\dfrac{r^2}{2a^2}$; for $\cos a$, $1-\dfrac{a^2}{2a^2}$; and for $\sin \frac{1}{2}\rho$, $\sin a$, $\dfrac{\rho}{2a}$, $\dfrac{a}{2a}$ respectively, a being the radius of sphere, and retaining only the terms of the second order in the curvature of the sphere, the locus reduces to the circle $4r^2-\rho^2=0$, and the curve $\rho^2=4(r^2-a^2\sin^2\theta)$, the pedal of a conic section having O as centre.

II. *Solution by the* PROPOSER.

In the figure, let CA, RB be small circles of the sphere drawn according to the conditions. The other lines, joining different points of the figure, represent arcs of great circles. Let PC measured on a great circle be denoted by ρ, the angle PCR by θ, the radius of the circle BR by r, the radius of the circle AC by a; then



$$\cos \mathrm{DP} = \cos a \cos 2r + \sin a \sin 2r \cos \mathrm{DAP},$$
$$\cos \mathrm{DP} = \cos a \cos \rho + \sin a \sin \rho \cos(\pi-\theta).$$

And since $\qquad \mathrm{DAB} = \mathrm{DCB} = \pi - \mathrm{BCP} - \theta,$

$$\sin a \sin 2r(\sin\theta \sin \mathrm{BCP} - \cos\theta\cos \mathrm{BCP}) = \cos a(\cos\rho - \cos 2r) - \sin a \sin\rho\cos\theta;$$

also $\qquad \cos \mathrm{BCP} = \dfrac{\cos r}{\sin r}\,\dfrac{1-\cos\rho}{\sin\rho};$

therefore $\sin a \sin 2r \left\{ \sin\theta \left(1 - \dfrac{\cos^2 r}{\sin^2 r}\,\dfrac{(1-\cos\rho)^2}{\sin^2\rho}\right)^{\frac{1}{2}} - \cos\theta\,\dfrac{\cos r}{\sin r}\,\dfrac{1-\cos\rho}{\sin\rho} \right\}$

$$= \cos a(\cos\rho - \cos 2r) - \sin a \sin\rho\cos\theta.$$

But $\quad \sin a \sin 2r \left(1 - \dfrac{\cos^2 r}{\sin^2 r}\,\dfrac{(1-\cos\rho)^2}{\sin^2\rho}\right)^{\frac{1}{2}} = \dfrac{2\sin a \cos r}{\cos\frac{1}{2}\rho}(\cos^2\tfrac{1}{2}\rho - \cos^2 r)^{\frac{1}{2}};$

therefore $\quad \dfrac{2\sin a\cos r}{\cos\frac{1}{2}\rho}\sin\theta(\cos^2\tfrac{1}{2}\rho - \cos^2 r)^{\frac{1}{2}} = 2\cos a(\cos^2\tfrac{1}{2}\rho - \cos^2 r)$

$$+ 2\sin a\cos^2 r \tan\tfrac{1}{2}\rho\cos\theta - 2\sin a\sin\tfrac{1}{2}\rho\cos\tfrac{1}{2}\rho\cos\theta$$
$$= 2(\cos^2\tfrac{1}{2}\rho - \cos^2 r)(\cos a - \sin a\tan\tfrac{1}{2}\rho\cos\theta).$$

The locus includes the small circle $(\cos^2\frac{1}{2}\rho - \cos^2 r)^{\frac{1}{2}} = 0$. Dividing this factor out, we get

$$\sin a\cos r\sin\theta = (\cos^2\tfrac{1}{2}\rho - \cos^2 r)^{\frac{1}{2}}(\cos a\cos\tfrac{1}{2}\rho - \sin a\sin\tfrac{1}{2}\rho\cos\theta) = 0,$$

whence we obtain the form given.

Neglecting quantities of a higher order than the second except as to θ, we find $$\rho^2 = 4r^2 \cos^2\theta + 4\,(r^2 - a^2)\sin^2\theta,$$ representing a central pedal of a conic section.

2023. (Proposed by W. A. Whitworth, M.A.)—If n be a prime number,

$$\lfloor\!\underline{\,n\,} \left\{ \frac{1}{n} - \frac{1}{n-1} + \frac{1}{\lfloor\!\underline{\,2\,}(n-2)} - \frac{1}{\lfloor\!\underline{\,3\,}(n-3)} + \ldots\ldots + \frac{1}{\lfloor\!\underline{\,n-1\,}} - \frac{1}{\lfloor\!\underline{\,n\,}} \right\}$$

is a positive integer of the form $\mathrm{M}(n) - 2$.

Solution by R. W. Genese.

It is obvious that, n being prime, all the terms of expression except $n\left\{\frac{1}{n} - \frac{1}{\lfloor\!\underline{\,n\,}}\right\}$ are integers and multiples of n.

We are left to show that $\lfloor\!\underline{\,n-1\,} - 1$ is of the form $\mathrm{M}(n) - 2$, that is that $\lfloor\!\underline{\,n-1\,}$ is of form $\mathrm{M}(n) - 1$. This is, in fact, Wilson's Theorem.

Next, we notice that the third term is greater than the fourth, the fifth than the sixth, and so on; for

$$\lfloor\!\underline{\,r+1\,} \left\{ n - (r+1) \right\} > \lfloor\!\underline{\,r\,}\,(n-r),$$

if $$(r+1)(n-r-1) > n-r, \quad \text{or} \quad r(n-r) > r+1,$$

which is always the case except when $n = r+1$, and this does not occur in our calculation.

Again, $$\text{sum of first two terms} = -\frac{1}{n(n-1)},$$

$$\text{sum of second two terms} = \frac{2n-7}{6\,(n-2)(n-3)};$$

the latter overcomes the former so long as $n > 4$, and balances it when $n = 4$.

Finally, therefore, the whole series is positive when $n > 3$; and it is clearly, on inspection, positive when $n = 3$.

C. F. Hodgson & Son, Gough Square, Fleet Street.

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